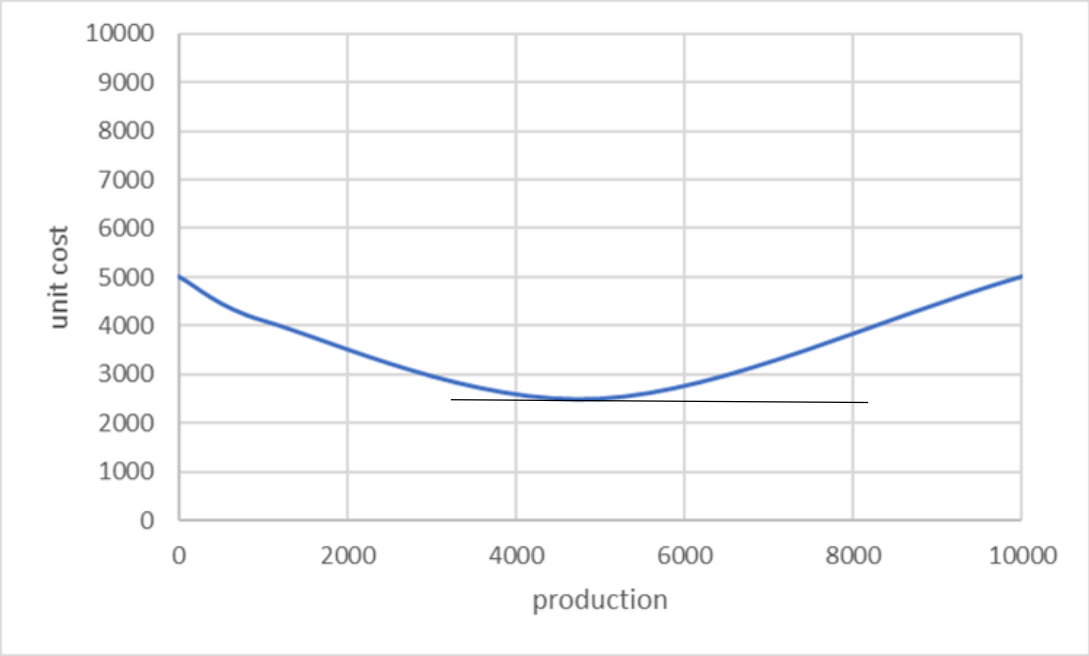
Math refresher for PPLE students

4. Finding extrema and total derivatives

Extrema

- Extrema are simply maximums of minimums of the value of a function
- A line has no minimum or maximum, so it has no extrema.
- A quadratic function may have a minimum or maximum:

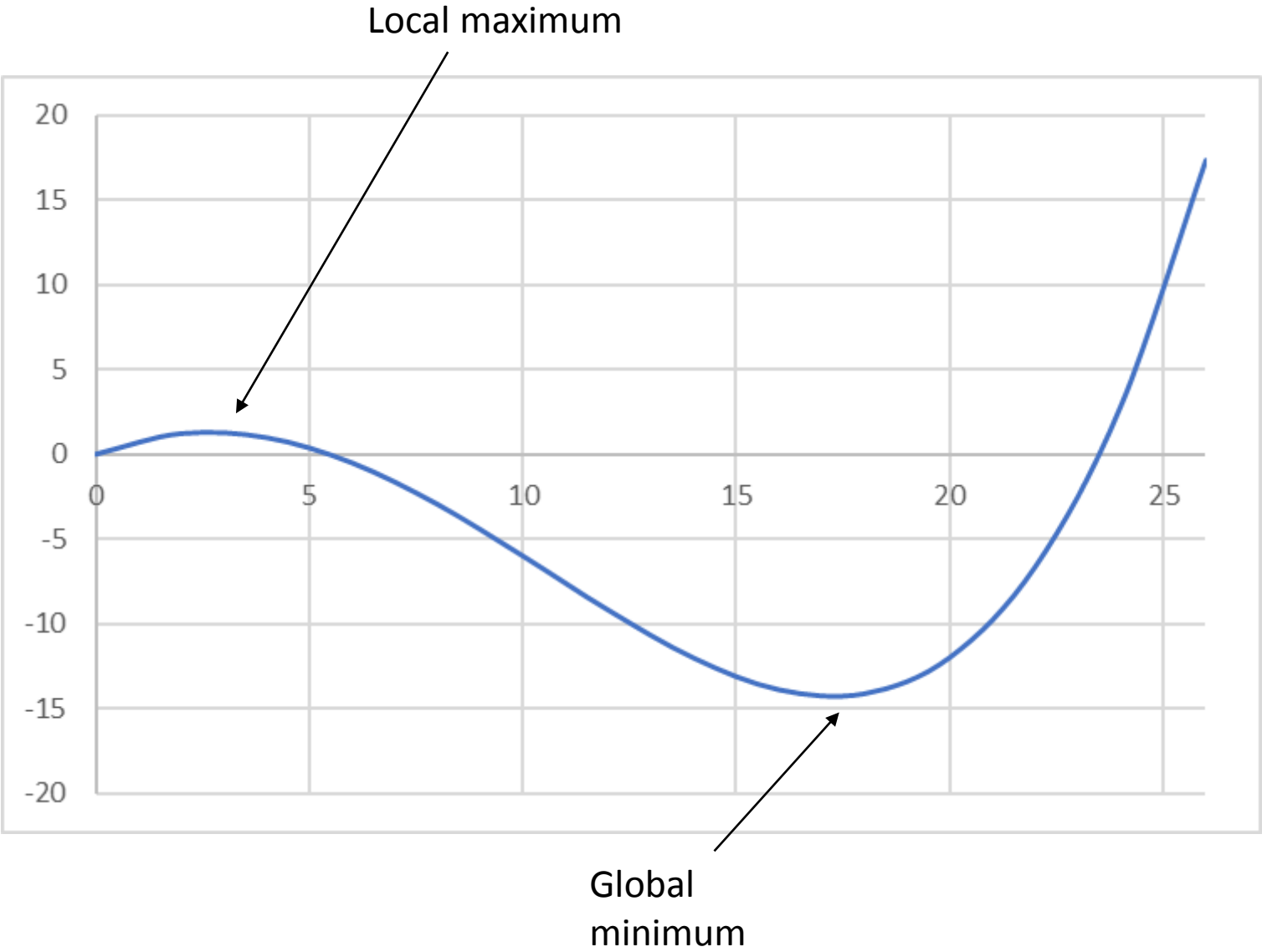


The minimum here is 2500 at

x=50000

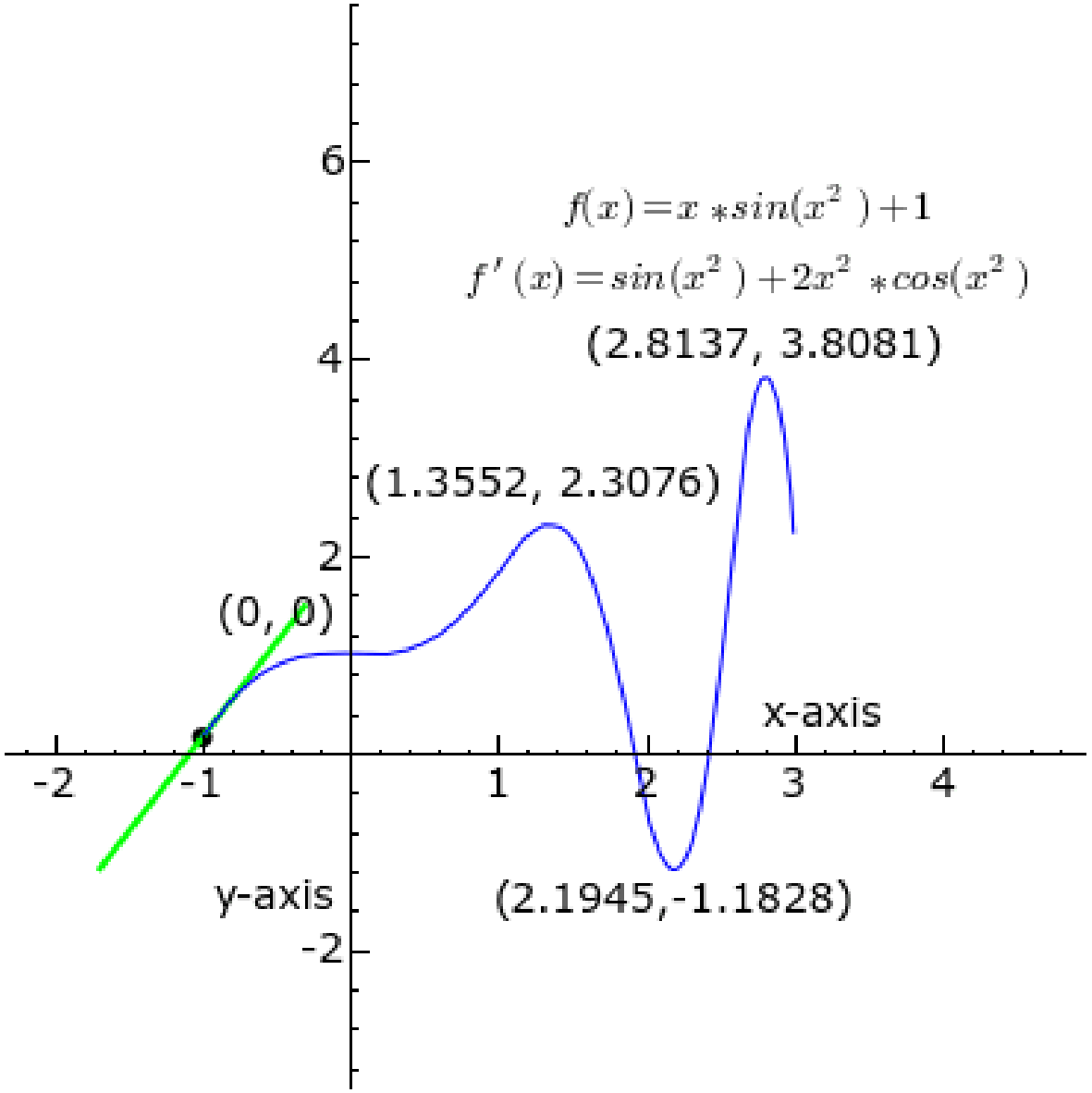
Extrema

- You can have mutliple extrema, a peak or trough may be local or global.
- The highest (lowest) extrem values are referred to as global.
- Here the function will grow infinitely after the minimum, so the minimum is the lowest possible value of it: it is a global minimum.
- The maximum is just local, though because at high x the y goes to infinity.



Finding an extrema

- At a maximum or minimum point of a function, the first derivative of a function is zero.
- Logically you can only climb until you reach the peak (positive slope).
- After the peak you start to descend (negative slope).
- So at the peak your slope must be zero. First-order condition of an extrema.



Finding the minimum

- Do you remember the average cost function of the car factory?

$$y = 5000 - x + 0.0001x^2$$

- How many cars should I produce to operate at the lowest possible average cost?

$$y' = 0.0002x - 1 = 0 \rightarrow x = 5000$$

- What is the per unit cost at this output?

$$y(x = 5000) = 5000 - 5000 + 0.0001(5000)^2 = 2500$$

Finding optimum consumption

- I have the following utility function, where x and y are two different products:

$$U = \ln(x) + \ln(y)$$

- I know that I can spend only 50 euro on them, and x costs 1 euro, y costs 2 euro (budget cosntraint).

$$50 = x + 2y$$

- How much of x and y should I consume to maximize my utility?

Finding optimum consumption

- Express y from the budget contraint:

$$y = 25 - 0.5x$$

- Substitute the budget constraint into the Utility function:

$$U = \ln(x) + \ln(y) = \ln(x) + \ln(25 - 0.5x)$$

- Now, the rest is easy. Let us take the first derivative of the utility function and make it equal to zero:

$$U' = \frac{1}{x} - \frac{0.5}{25 - 0.5x} = 0 \rightarrow x = 25$$

- From this:

$$y = 25 - 0.5 \cdot 25 = 12.5$$

Total derivative

- Total derivative is useful when we have multivariate functions. For example:
$$f(x,z) = ax + bz$$
- We can either assume that only x or z changes and the other is fixed. This is partial derrivative:
$$\frac{\partial f(x,z)}{\partial x} = a, \frac{\partial f(x,z)}{\partial z} = b$$
- But what if both changes by dx and dz? Then we have a total derivative:
$$df(x,z) = \frac{\partial f(x,z)}{\partial x} dx + \frac{\partial f(x,z)}{\partial z} dz = adx + bdz$$

Total derivative

- That is, the total derivative is:

change in the function's value=(derrivative with repect to x) X (change in x)+
+(derrivative with repect to z) X (change in z)

$$\mathrm{df(x,z)}=\frac{\partial f(x,z)}{\partial x}dx+\frac{\partial f(x,z)}{\partial z}dz$$

Use of total derivative

- I can only eat apples and cookies. One apple had the total calorie of 50 kcal and one cookie has 300 kcal.
- I can only eat 1500 kcal per day.
- If I eat one cookie, how much less apple should I eat to keep myself at the limit?
- the calorie constraint is: $1500 = 50a + 300c$
- The objective is to keep the value of the function constant. Let us take the total derivative:

$$0 = 50 \cdot da + 300 \cdot dc$$

Use of total derivative

- Let us express the ratio of the change in appes and cookies:

$$0 = 50 \cdot da + 300 \cdot dc \rightarrow da = -6dc \rightarrow \frac{da}{dc} = -6$$

- The ratio is -6, so if I eat one additonal cookie, I must eat 6 apples less in order to keep the calorie limit. This is the trade-off between apples and cookies.

Use of total derivative

- I can only eat apple and cookies (a and c). My utility function is as follows:

$$U = 2\ln(c) + \ln(a)$$

- Where c is the number of cookies and a is the number of apples I eat.
- How much apple am I willing to give up for one additional cookie?

$$0 = \frac{2}{c}dc + \frac{1}{a}da \rightarrow \frac{da}{dc} = -2\frac{a}{c}$$

Use of total derivative

- So my willingness to trade the two goods depends on the actual ratio of consumption.

$$\frac{da}{dc} = -2\frac{a}{c}$$

- When I eat equal quantity of apples and cookies, I am willing to trade 2 apples for one cookie .
- When I eat 5 times as many apples than cookies, then I am willing to give up 10 apples for one cookie.
- I will be willing to trade them at 1:1 ratio if I eat exactly twice as many cookies than apples.