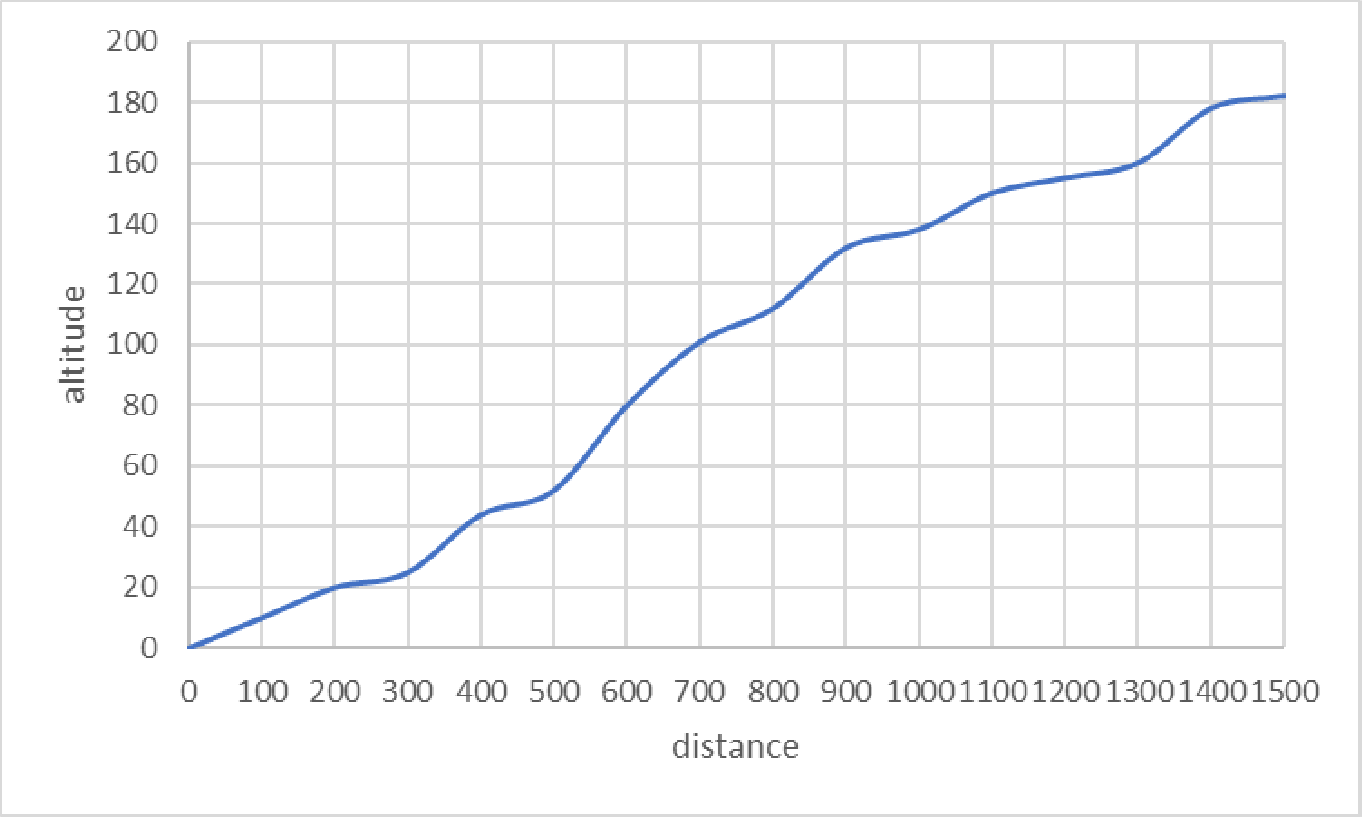
Math refresher for PPLE students

3. Derivatives

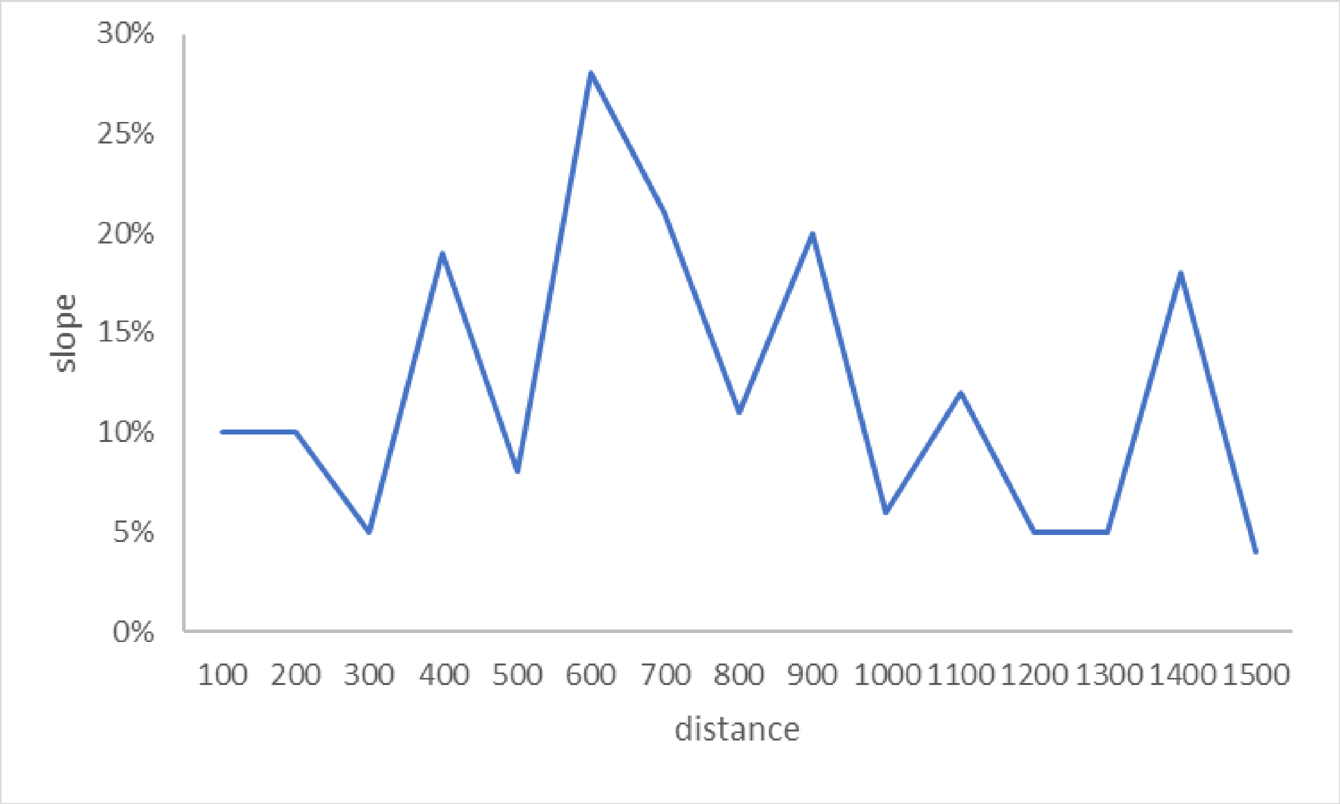
A heuristic approach 1

- In the graph to the right you can see a hillside.
- I start to climb the hill along a path. The more I walk along the path (the more distance I cover) the higher I get.
- This is a function, since by knowing how much I walked, I also learn something about my altitude.



A heuristic approach 2

- What is the slope of the hillside?
- I can say that the slope in general is the altitude I gain by advancing along the path.
- But you can see that this slope is changing!
- In the first 100 meter I go up 10 meters, while between 1100th and 1200th meter only 5 meters. (10% vs 5% slope).
- The slope seems to change with the distance: it is a function of it.



Slope

- Generally the slope of a function is going to be the ratio between the change in the value of function and the change in the explanatory variable or the average rate of change in y over an interval of x.

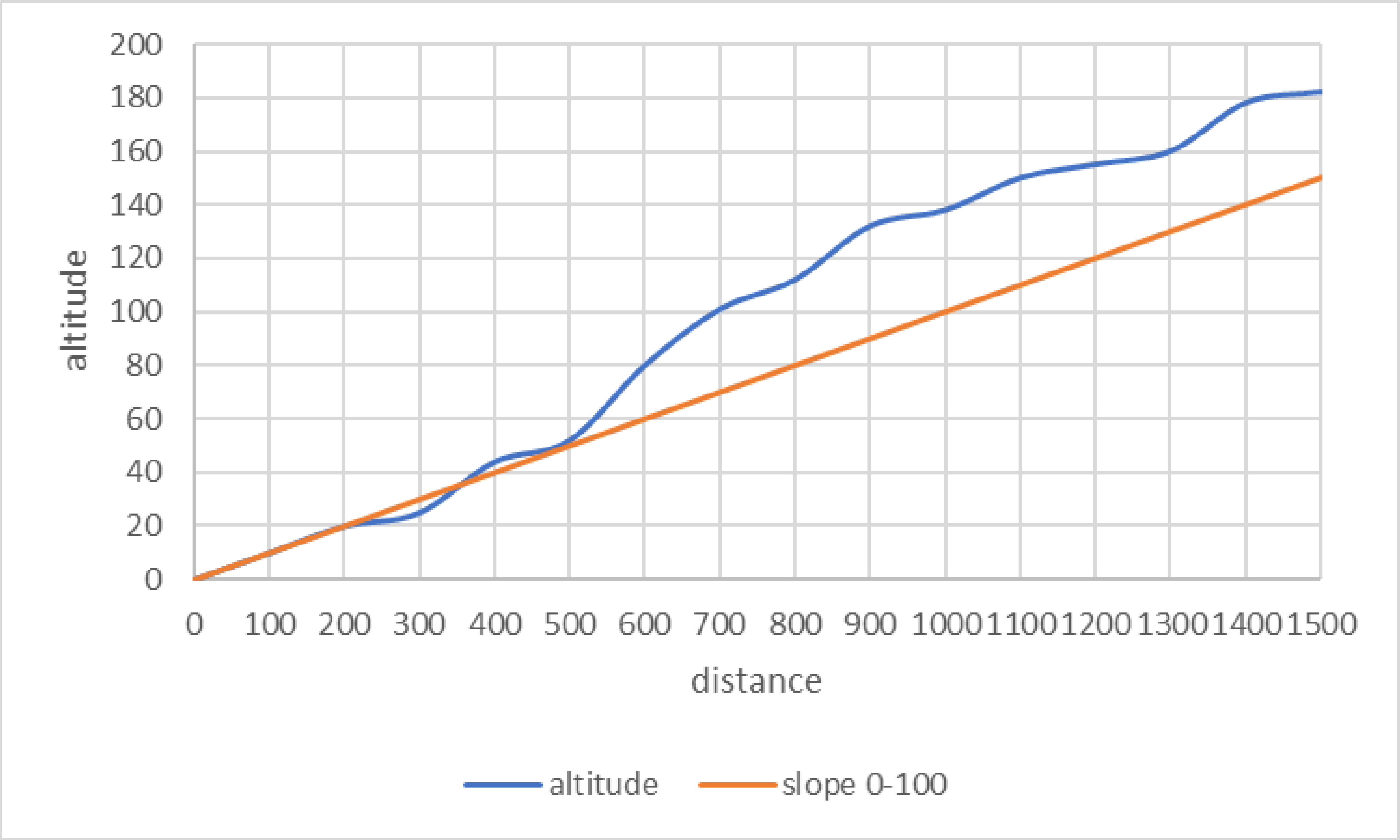
$$d = \frac{\Delta y(x)}{\Delta x} = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

- When I start my walk, in the first 100 meter I gain 10 meters:

$$d_{0-100} = \frac{10-0}{100-0} = 0.1$$

- But I can also calculate a slope for the whole hillside:

$$d_{0-1500} = \frac{182-0}{1500-0} = 0.121$$



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distance
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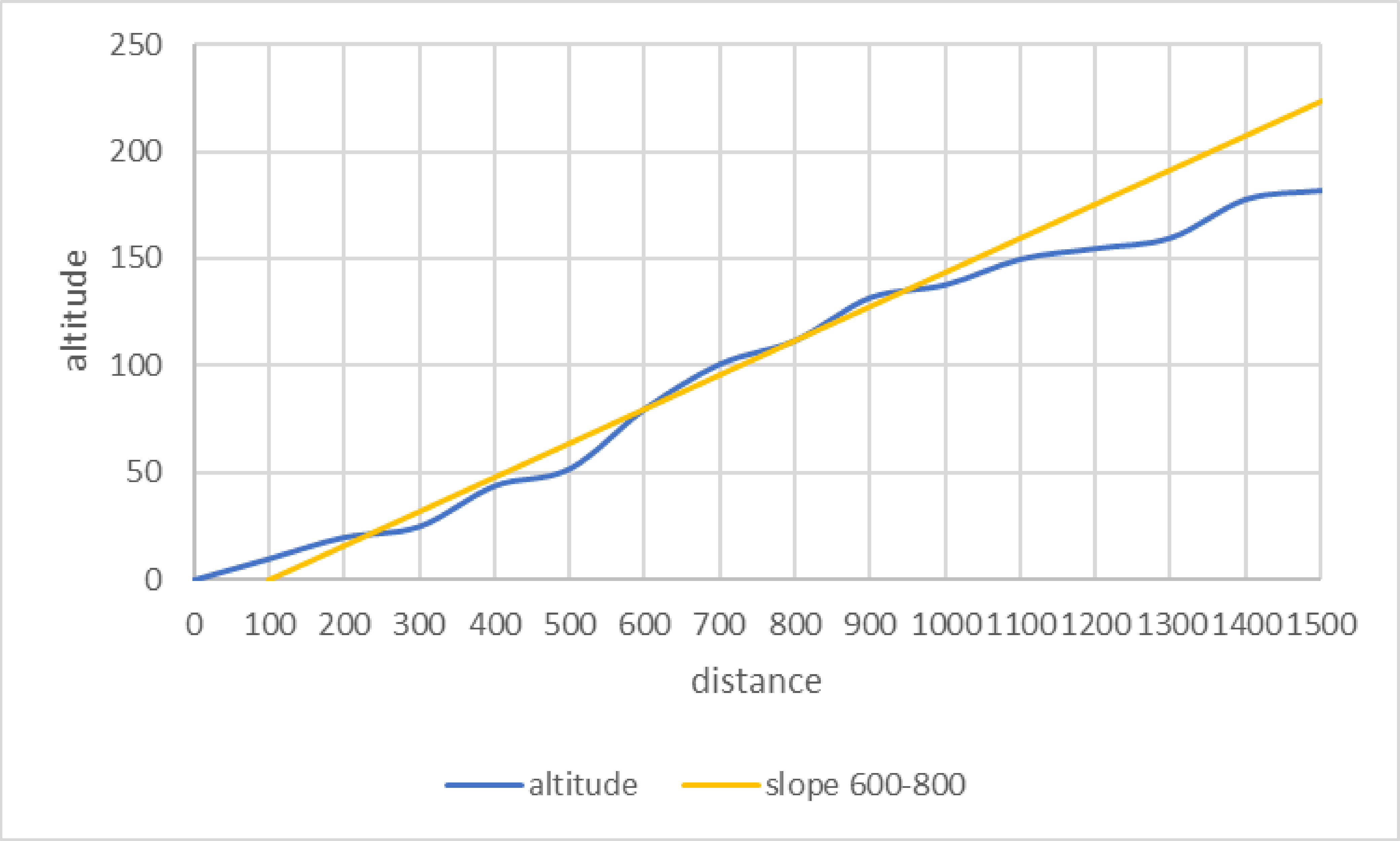
slope 0-100



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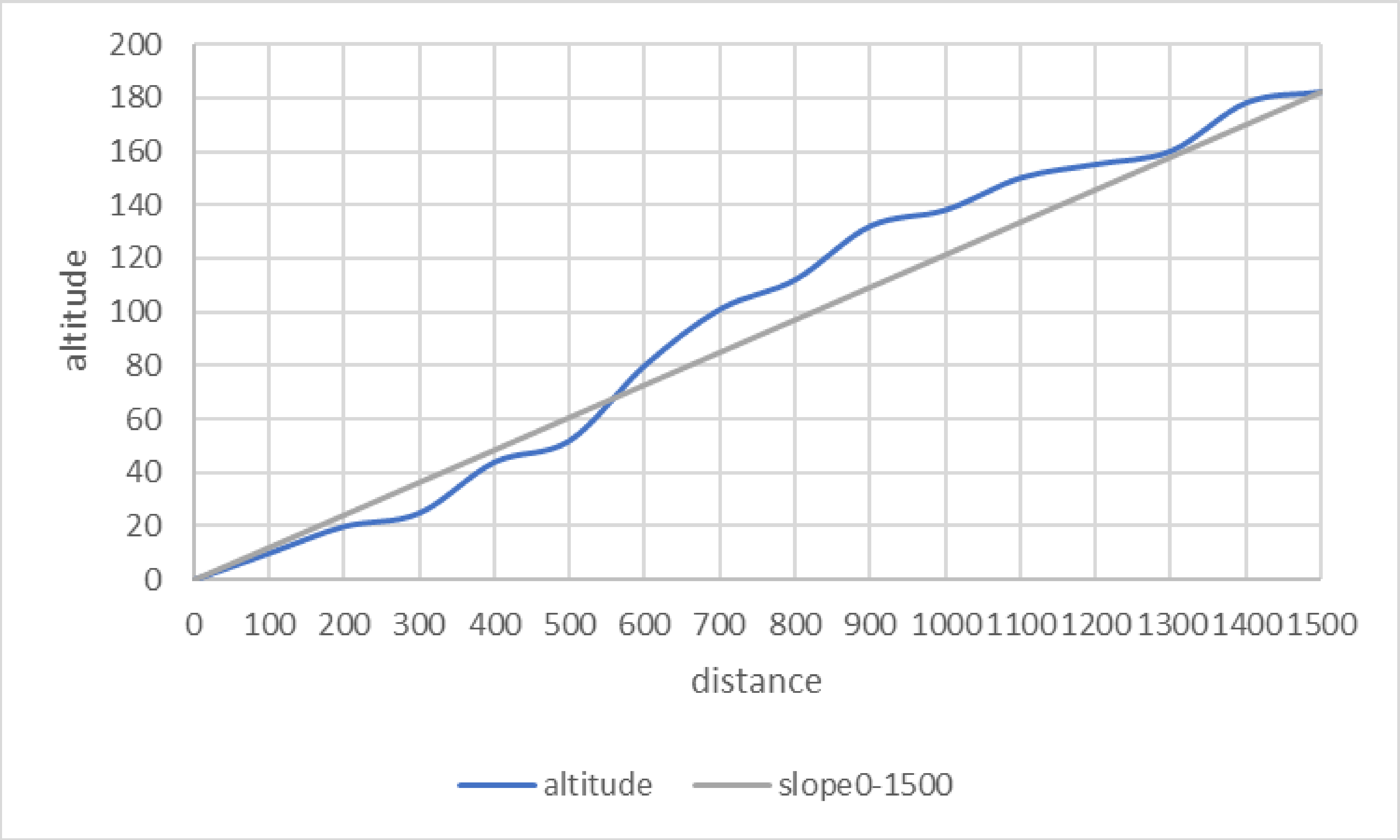

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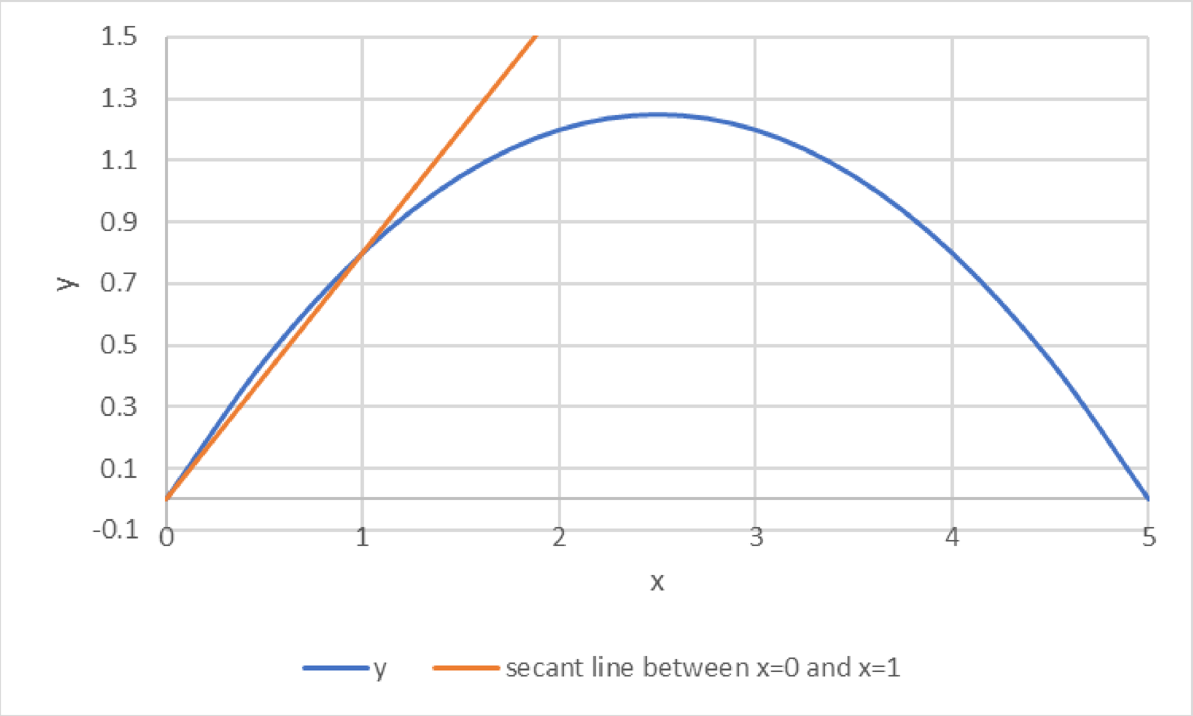
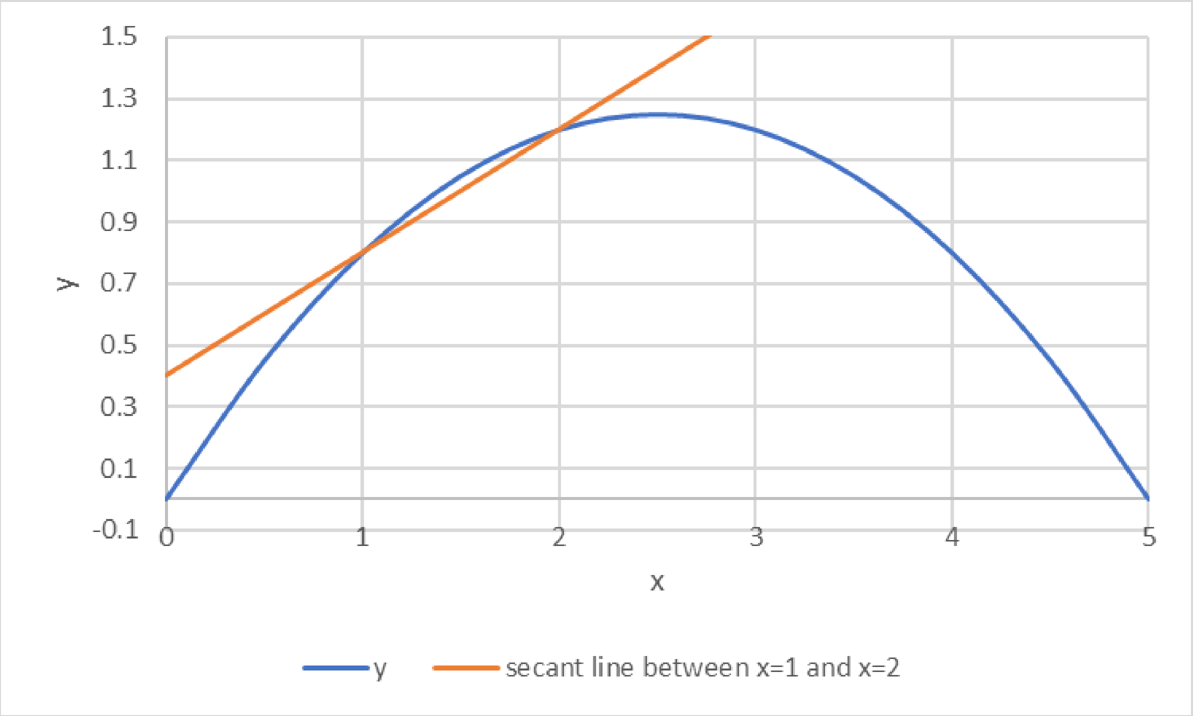
slope0-1500

The secant line

- When we use a function then we call any straight line that connects any two points of the function a secant line. For example if we have quadratic function, then the secant line between x=0 and x=1 is different from the secant line between x=1 and x=2:





Motivation for using the slope of the tangent line

- Looking at the sope of the secant line makes sense if the variable x takes only discrete values. Consider car production. I can increase the production by a whole number only since 0.5 car does not exist.
- Then Δx from the formula

$$d = \frac{\Delta y(x)}{\Delta x} = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

can only be a whole number.

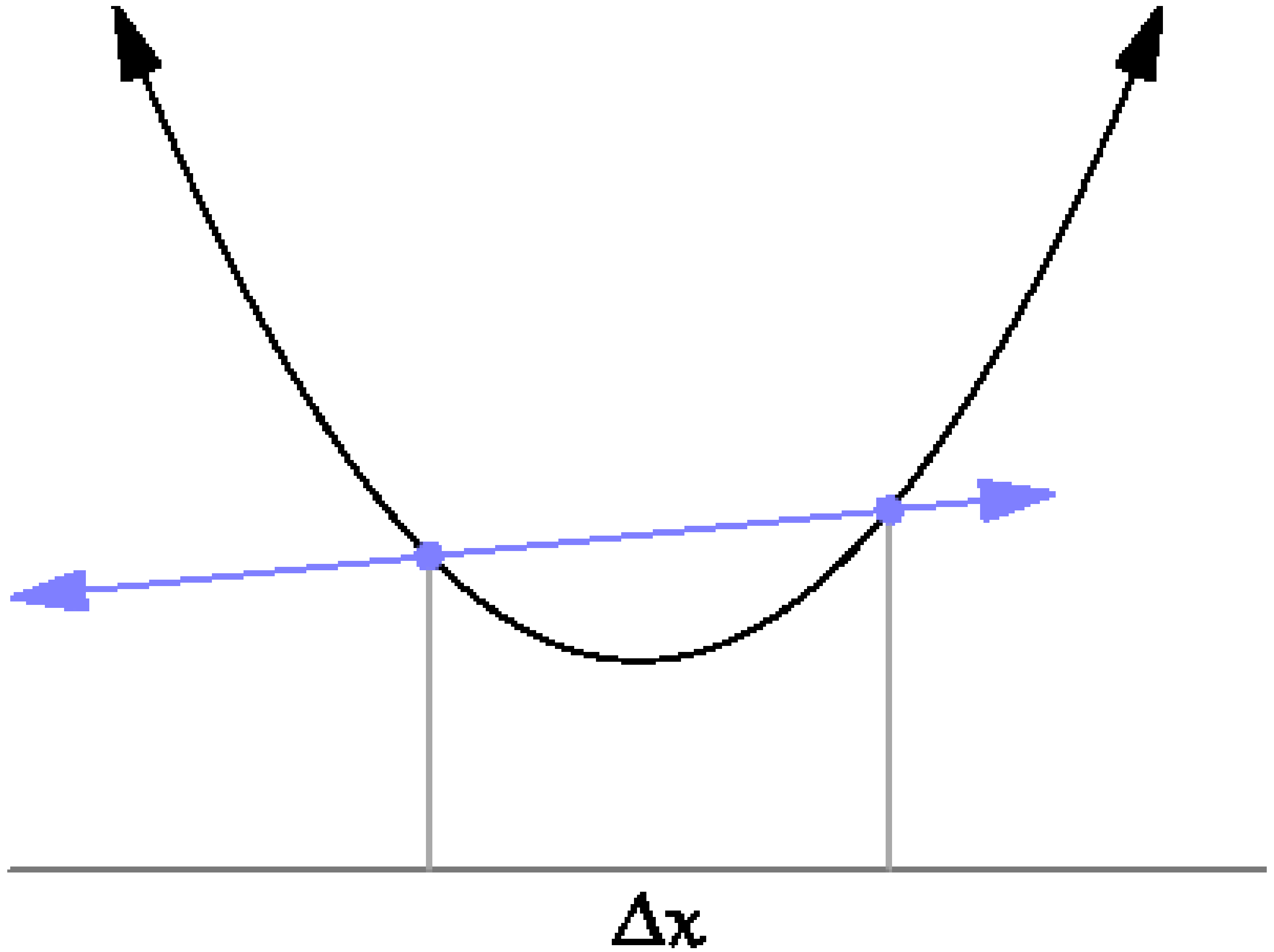
- But what if x may take any values? (this is called continous variable) Then Δx may be smaller the one, actually infinitely small. When we decrease Δx toward zero the secant line becomes a tangent line.

The tangent line

- What we do is that from the formula:

$$d = \frac{\Delta y(x)}{\Delta x} = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

- We start to decrease the distance between x_1 and x_0. As this distance approaches zero, we get a tangent line and its slope can be interpreted as the slope belonging to a single point at x, since the difference between x_1 and x_0 becomes negligible.
- The tangent line just touches the function at one point but does not intersect it.
- The difference is just like between speed and velocity. Speed is measured as the distance covered in a unit of time (what you calculate later as average speed over a distance), while velocity is instantaneous (what your speedometer shows at the moment).


Δx

The derivative

- The first derivative of difference of function f(x) is a function that links the values of variable x with the slope of their respective tangent line.
- It is denoted in many different ways:

$$f'(x) \qquad y' \qquad \frac{\partial f(x)}{\partial x} \qquad \frac{\partial y}{\partial x}$$

- But they all mean the same: a function that gives you the slope of the tangent line for every single x value.

Rules of derivation

- The derivative of the sum of functions is the sum of derivatives:

$$y = f(x) \pm g(x)$$

$$y' = f'(x) \pm g'(x)$$

- The derivative of the product of functions is:

$$y = f(x) \cdot g(x)$$

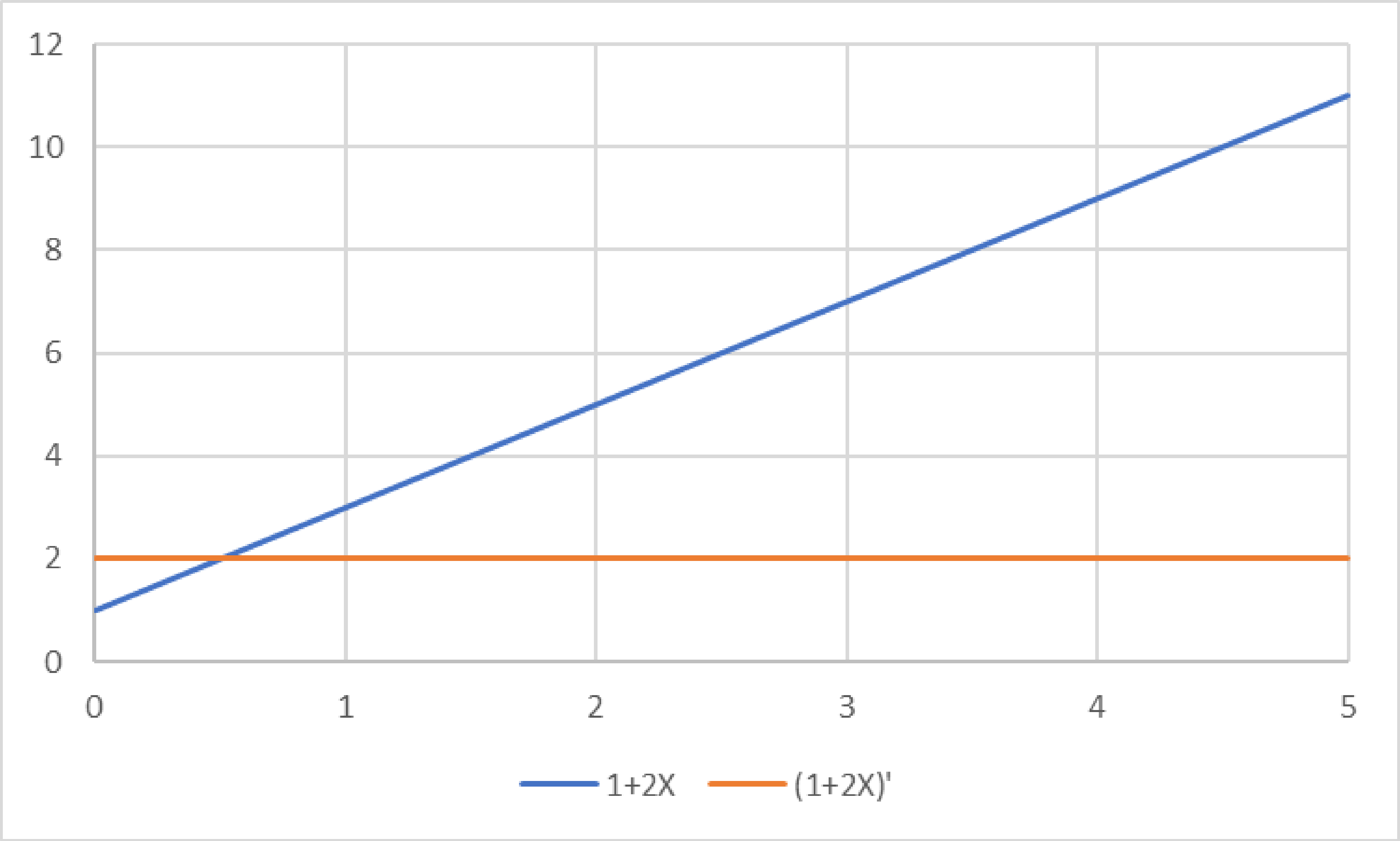
$$y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- The derivative of a composite function is:

$$y = f(g(x))$$

$$y' = f'(g(x)) \cdot g'(x) = \frac{\partial f(g(x))}{\partial g(x)} \cdot \frac{\partial g(x)}{\partial x}$$

The first derivative of a line is a constant



The first derivative of a line is a constant

- Since the line has a constant slope, the first derivative must be a constant too.

$$y = a + bx$$

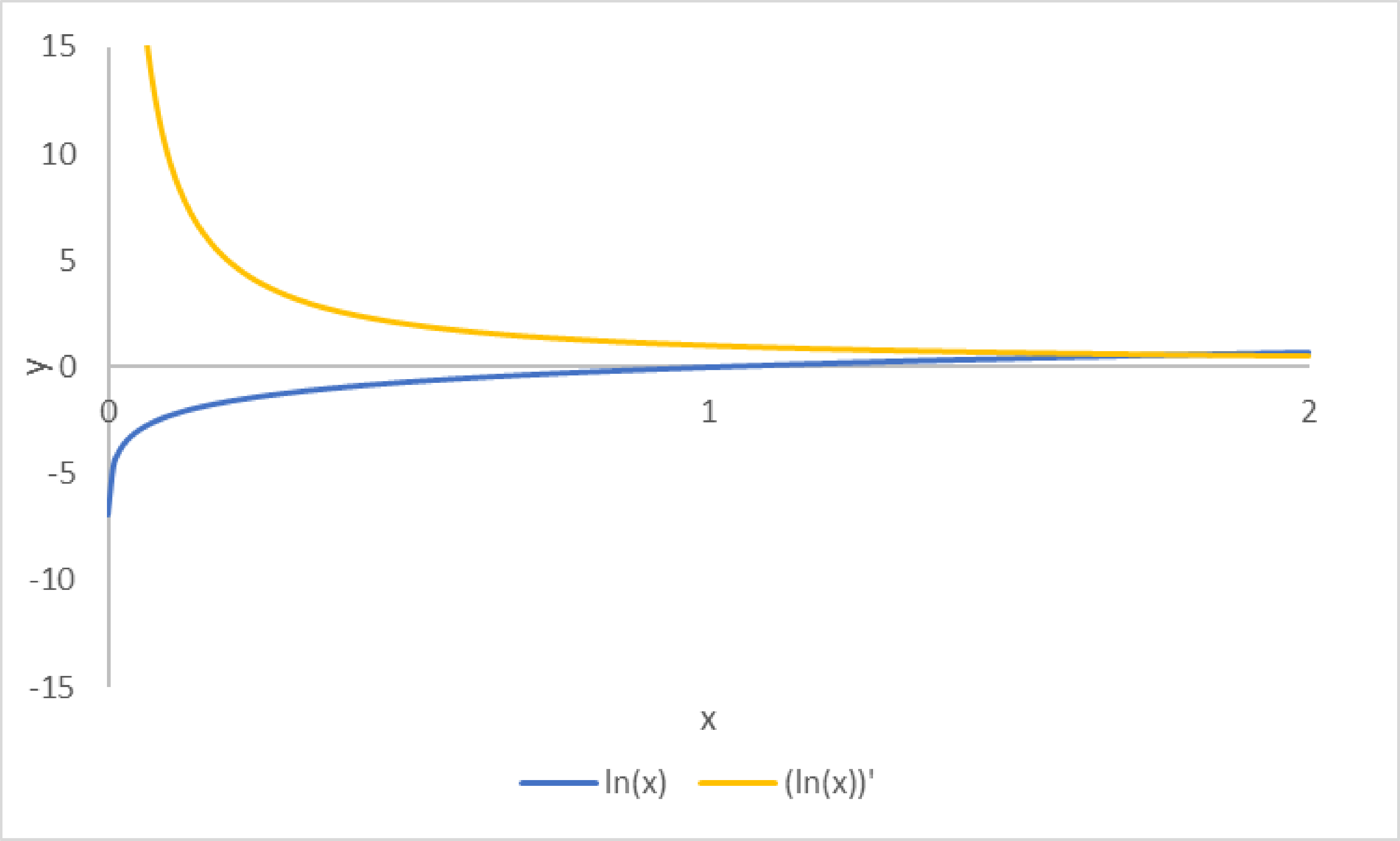
$$y' = b$$

- Logially, if you had only a constant, then it has no slope, so:

$$y = a$$

$$y' = 0$$

The first derivative of ln(x) is 1/x



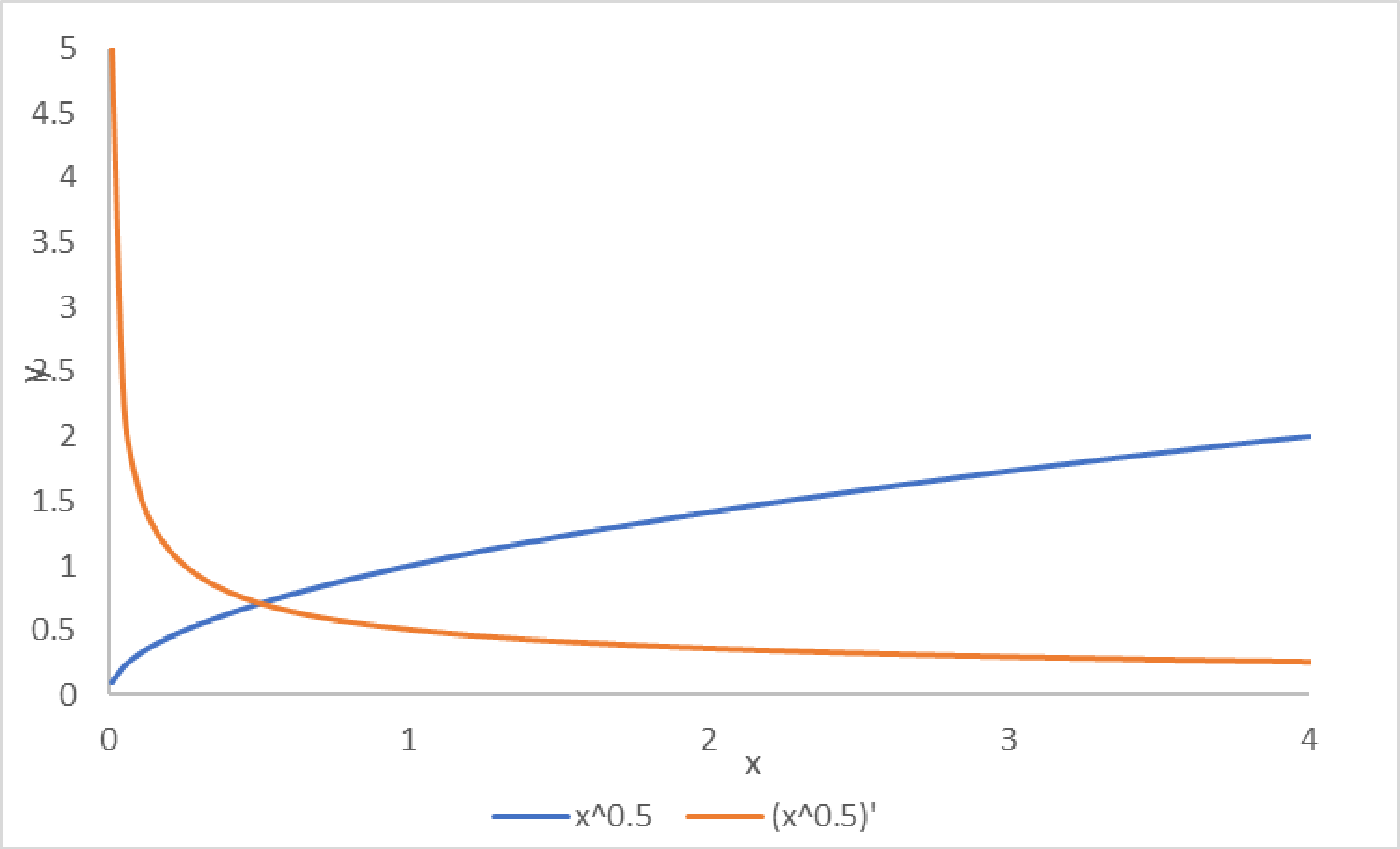
The first derivative of ln(x) is 1/x

- For the ln(x):

$$y = \ln(x)$$

$$y' = \frac{1}{x}$$

The first derivative of a power function is a power function with different coefficients

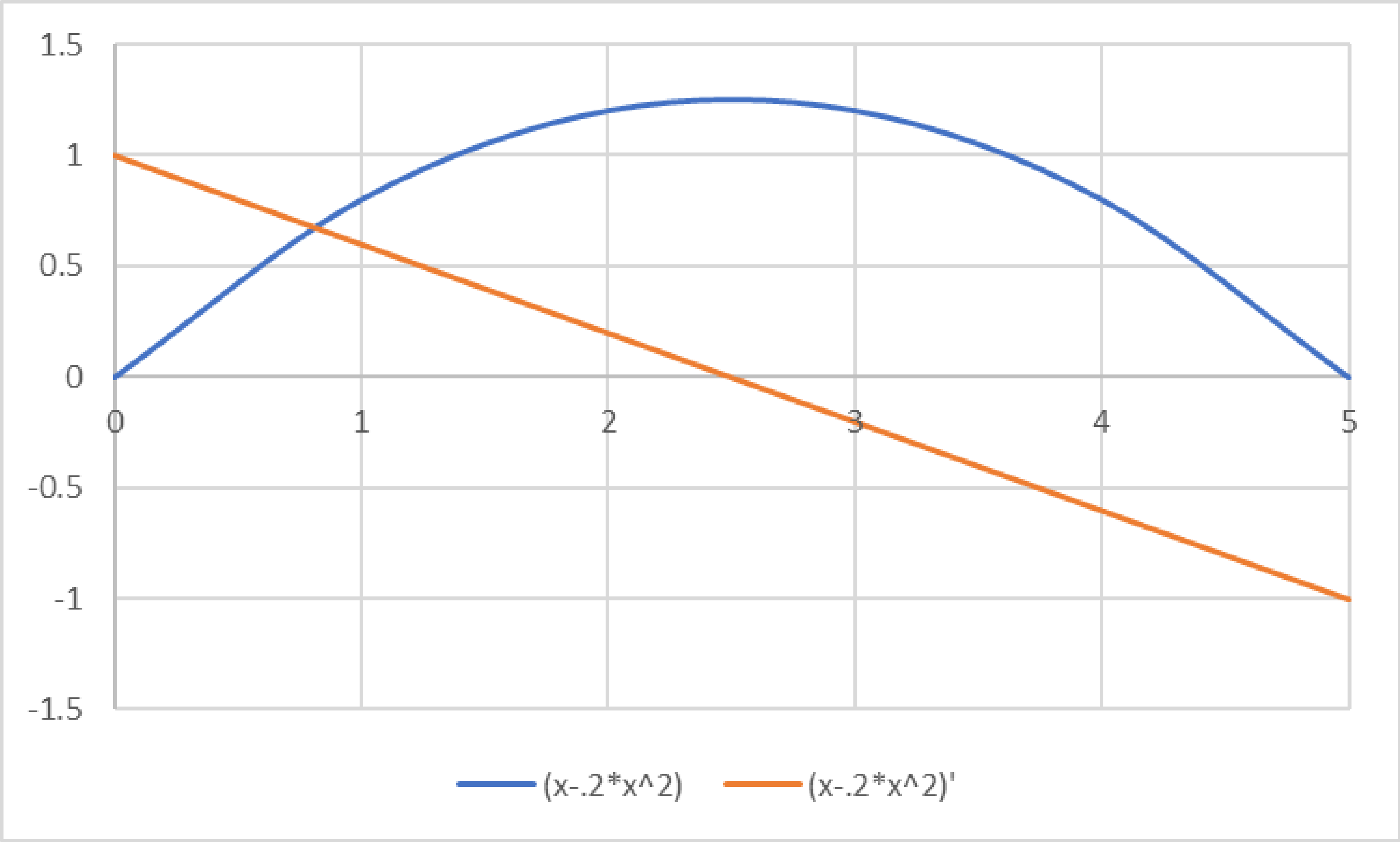


First derivative of a power function

$$y = ax^{b}$$

$$y' = bax^{b-1}$$

The first derivative of a quadratic function is a line



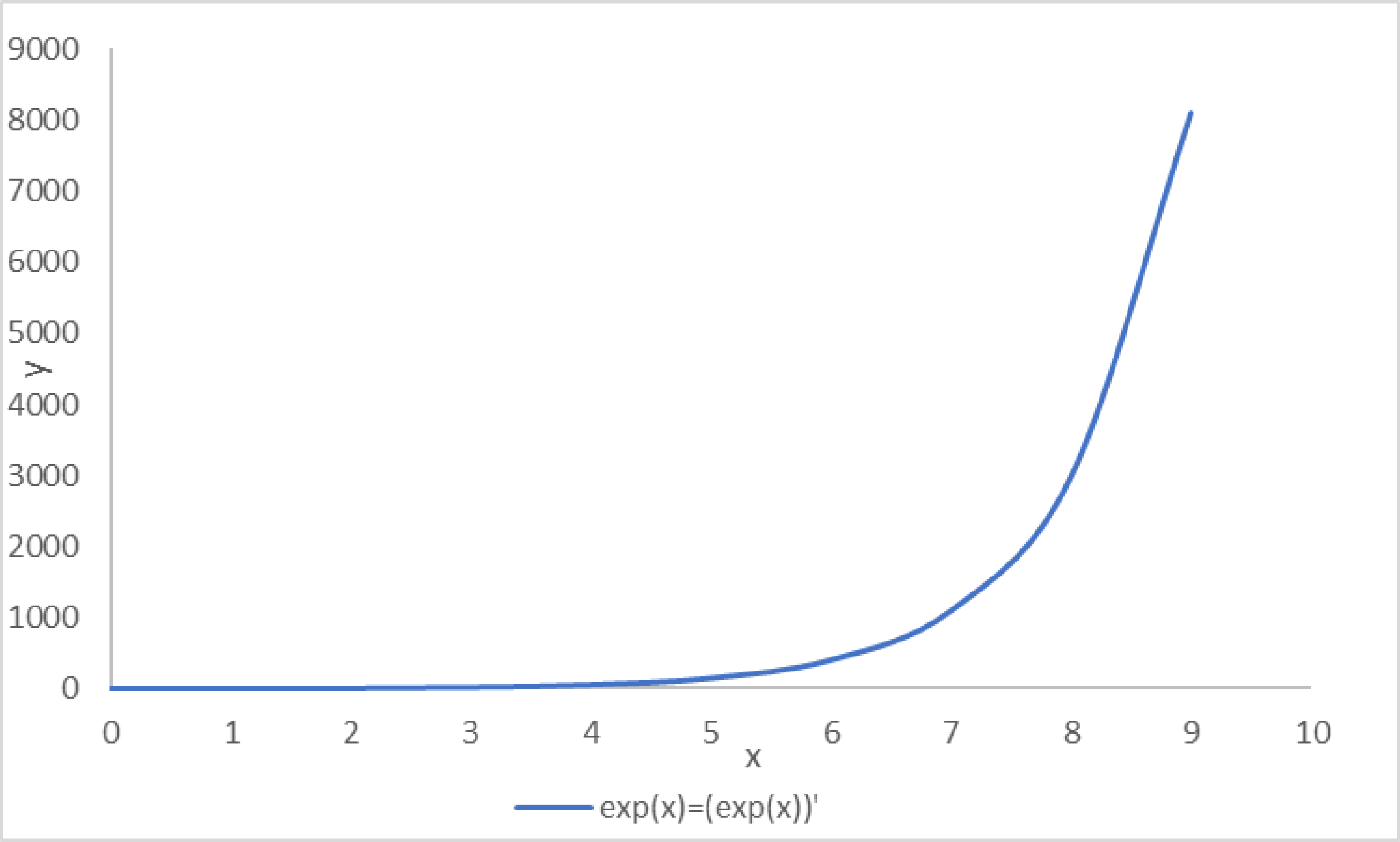
First derrivative of a quadratic function

- Here we can use what we learned about power functions and the summation rule:

$$y = a + bx + cx^2$$

$$y' = b + 2cx$$

The first derivative of an exponential function is an exponential function



Exponential function

$$y = ae^{x}$$

$$y' = ae^{x}$$

Practice

- What is the derivative of the following functions:

$$y = 2 + 3x \qquad y = e^{a+bx}$$

$$y = 2x^3 \qquad y = 3 - 0.5x + 0.05x^2$$

$$y = 0.5x^{0.5} \qquad y = x^{0.3} z^{0.7}$$

$$y = \ln(x+1)$$

Solutions

$$y = 2 + 3x \to y' = 3$$

$$y = 2x^3 \to y' = 3 \cdot 2x^{(3-1)} = 6x^2$$

$$y = 0.5x^{0.5} \to y' = 0.5 \cdot 0.5x^{(0.5-1)} = 0.25x^{-0.5} = \frac{1}{4\sqrt{x}}$$

$$y = \ln(x+1) = \frac{1}{x+1} \cdot 1 = \frac{1}{x+1}$$

Solutions

$$y = e^{a+bx} \rightarrow y' = be^{a+bx}$$

$$y = 3 - 0.5x + 0.05x^2 \rightarrow y' = -0.5 + 0.1x$$

$$y = x^{0.3}z^{0.7} \rightarrow y' = 0.3x^{-0.7}z^{0.7} = 0.3\frac{y}{x}$$