Advanced time-series analysis (University of Lund, Economic History Department)

30 Jan-3 February and 26-30 March 2012

Lecture 10 Vector Error Correction (VEC): Johansen technique of cointegration testing, empirical applications.

10.a Multiple cointegrating vectors

With the single-equation time-series techniques we could only estimate a single cointegrating vector. Let us take an example: we have three I(1) variables, x_t, y_t, and z_t that are thought to be cointegrated of order one. If you have a single cointegrating vector (where the coefficient of y is normalized to one: $\boldsymbol{\beta} = (1, -\beta_0, -\beta_1, -\beta_2)'$

Then you can write the equilibrium value of y_t as follows:

$$\tilde{y}_t = \beta_0 + \beta_1 x_t + \beta_2 z_t$$

and the deviation from it as:

$$u_t = y_t - \tilde{y}_t$$

Now the error-correction model is:

$\Delta y_t = \theta_{10} + \theta_{11}\Delta x_t + \theta_{12}\Delta z + \alpha_1 u_{t-1} + e_{1t}$, here we expect that α_{1t} is negative and is between 0 and -4, so that you have a return to the equilibrium value of y. (Now the adjustment coefficient is denoted by alpha, and the short-run coefficients are denoted by theta; this is different than in the lecture notes no. 6, where we used gamma for the adjustment coefficients and alpha's for the immediate effects, but the current one seems the standard (most common) notation for a VEC.)

You could actually rewrite this equation with the change of any other cointegrated variables at the left-hand side:

$\Delta x_t = \theta_{20} + \theta_{21}\Delta y_t + \theta_{22}\Delta z + \alpha_2 u_{t-1} + e_{2t}$, this also makes sense, but since u_t is still the deviation of y_t form its equilibrium value and not a deviation of x_t from its own equilibrium value, α_{2t} does not have to be negative. Actually what this coefficient is going to show you is, how x_t reacts if y_t deviates from its equilibrium: if this relationship is symmetrical in the sense that both x_t and z_t are going to adapt, you expect that α_{2t} is going to be different from zero. Let us assume that β_1 is a positive number! In this case, if u_t gets positive (y_t is above its equilibrium), x_t may also react to this by an increase an so removing some of the deviation. In this case you should obtain a positive value for α_{2t}.

Now, let us see substitute the cointegrating vector into the ECM:

$$\Delta y_t = (\theta_{10} - \alpha_1\beta_0) + \theta_{11}\Delta x_t + \theta_{12}\Delta z_t + \alpha_1 y_{t-1} - \alpha_1\beta_1 x_{t-1} - \alpha_1\beta_2 z_{t-1} + e_{1t}$$

$$\Delta x_t = (\theta_{20} - \alpha_2\beta_0) + \theta_{21}\Delta y_t + \theta_{22}\Delta z_t + \alpha_2 y_{t-1} - \alpha_2\beta_1 x_{t-1} - \alpha_2\beta_2 z_{t-1} + e_{2t}$$

$$\Delta z_t = (\theta_{30} - \alpha_2\beta_0) + \theta_{31}\Delta y_t + \theta_{32}\Delta x_t + \alpha_3 y_{t-1} - \alpha_3\beta_1 x_{t-1} - \alpha_3\beta_2 z_{t-1} + e_{3t}$$

Believe or not, we are already quite close to a VEC with a single cointegrating vector. I know that it is not a very popular suggestion but let us rewrite the above system of equations into matrix form:

$$\begin{pmatrix} 1 & -\theta_{11} & -\theta_{12} \\ -\theta_{21} & 1 & -\theta_{22} \\ -\theta_{31} & -\theta_{32} & 1 \end{pmatrix}\begin{pmatrix} \Delta y_t \\ \Delta x_t \\ \Delta z_t \end{pmatrix} = \begin{pmatrix} \theta_{10}-\alpha_1\beta_0 \\ \theta_{20}-\alpha_2\beta_0 \\ \theta_{30}-\alpha_3\beta_0 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}\begin{pmatrix} 1 & -\beta_1 & -\beta_2 \end{pmatrix}\begin{pmatrix} y_{t-1} \\ x_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{pmatrix}$$

or

$\mathbf{A\Delta Y_t} = \boldsymbol{\delta} + \boldsymbol{\alpha\beta}'\mathbf{Y_{t-1}} + \mathbf{e_t}$ very often the product $\boldsymbol{\alpha\beta}'$ is denoted by $\mathbf{\Pi}$.

Now we are going to look at the option when you have two cointegrating vectors:

$$\begin{pmatrix} 1 & -\theta_{12} & -\theta_{13} \\ -\theta_{21} & 1 & -\theta_{23} \\ -\theta_{31} & -\theta_{32} & 1 \end{pmatrix}\begin{pmatrix} \Delta y_t \\ \Delta x_t \\ \Delta z_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \end{pmatrix}\begin{pmatrix} 1 & -\beta_{12} & -\beta_{13} \\ 1 & -\beta_{22} & -\beta_{23} \end{pmatrix}\begin{pmatrix} y_{t-1} \\ x_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{pmatrix}$$

In standard form it would look:

$$\Delta y_t = \delta_1 + \theta_{12}\Delta x_t + \theta_{13}\Delta z_t + \alpha_{11}(y_{t-1} - \beta_{12}x_{t-1} - \beta_{13}z_{t-1}) + \alpha_{21}(y_{t-1} - \beta_{22}x_{t-1} - \beta_{23}z_{t-1}) + e_{1,t}$$
$$\Delta x_t = \delta_2 + \theta_{21}\Delta y_t + \theta_{23}\Delta z_t + \alpha_{12}(y_{t-1} - \beta_{12}x_{t-1} - \beta_{13}z_{t-1}) + \alpha_{22}(y_{t-1} - \beta_{22}x_{t-1} - \beta_{23}z_{t-1}) + e_{2,t}$$
$$\Delta z_t = \delta_3 + \theta_{31}\Delta y_t + \theta_{32}\Delta x_t + \alpha_{13}(y_{t-1} - \beta_{12}x_{t-1} - \beta_{13}z_{t-1}) + \alpha_{23}(y_{t-1} - \beta_{22}x_{t-1} - \beta_{23}z_{t-1}) + e_{3,t}$$

Of course, by a simple equation by equation estimation you could never separate the two vectors, so you could not estimate them. What you would estimate for the first equation is:

$$\Delta y_t = \delta_1 + \theta_{12}\Delta x_t + \theta_{13}\Delta z_t + (\alpha_{11} + \alpha_{21})y_{t-1} - (\alpha_{11}\beta_{12} + \alpha_{21}\beta_{22})x_{t-1} - (\alpha_{11}\beta_{13} + \alpha_{21}\beta_{23})z_{t-1} + e_{1,t}$$

and so on for the rest. As you can see, even if there were multiple cointegrating vectors, you could only estimate some kind of combination of them by a single equation method. So instead we prefer a vector approach, estimating a whole system and estimating matrix **Π** from that.

But first of all: what does it mean if you have multiple cointegrating vectors? When you have only a single one, you interpreted the existence of the cointegration by assuming that there was some kind of equilibrium relationship among your variables (a kind of common trend or co-movement) that did not allow them to wander off from this path indefinitely. Now you only need to assume that if you have k endogenous variables, you may have at most k-1 number of equilibrium relationships that exist and operate simultaneously. Let us take an example: the long-run movement of exchange rates can be explained by two mutually not-excluding theories: purchasing power parity (PPP) and uncovered interest rate parity (UIP).

PPP is based on the law of one price, that the purchasing power of a currency should be the same in all countries, i.e. spot exchange rates should adjust so, that the same good costs the same everywhere. That is: $P_t = S_t P_t^* \rightarrow \ln S_t = \ln P_t - \ln P_t^*$, where P_t and P_t^* are the price of the same good at home and abroad respectively, while S_t is the spot exchange rate expressed as domestic currency per one unit of foreign currency. This is clearly a mechanism that can govern long-run movement of foreign exchange rates.

The UIP is about investment decisions. Let us assume that you can invest 100$ in dollar in the USA at interest rate of 5%, or you can invest the same amount of money in euro at interest rate 4% but first you need to convert your money to euro at the current exchange rate of 1.2 $/€. So after the first year you either have 5$ or 0.04*100/1.2=3.33€. Which one is the better? It depends on what exchange rate we expect at the end of the year? If it is under 5/3.33=1.5 then it is better to stay with the in dollar denominated investment. If you expect the dollar depreciates more, and it gets above 1.5 \$/€ then you should rather go for the investment in euro. Obviously the equilibrium is when it does not matter in which currency you have your investments because one you bring it to the same currency, they have equal payoff.

$(1+i_t)=\frac{E_t S_{t+1}}{S_t}(1+i_t^*)$ where I and i* are the nominal interest rates on your investments in domestic and foreign currency respectively, and $E_t S_{t+1}$ is the expectation in t about the spot exchange rate in t+1. In order that the equality holds, if foreign interest rates rise with respect to the domestic ones, you expect that S is going to fall, that is, the domestic currency appreciates/foreign currency depreciates.

So: $E_t(\Delta \ln S_{t+1}) = i_t - i_t^*$, where we made us to the approximation: $\ln(1+x)\approx x$

These are two mechanisms that may both exist and affect the change in exchange rates. This can finally lead to two cointegrating vectors. Of course this is just a single example of many possibilities.

10.b From VAR to VEC

In the previous section we approached cointegration with the possibility of multiple cointegrating vectors departing from an Engle-Granger (single equation) method. Let us arrive at the same, but this time departing from a VAR!

Say, we have a VAR(2):

$$\mathbf{Y_t} = \boldsymbol{\delta} + \boldsymbol{\Theta_1}\mathbf{Y_{t-1}} + \boldsymbol{\Theta_2}\mathbf{Y_{t-2}} + \mathbf{e_t}$$

This can be rewritten in a Vector Error-Correction (VEC) form as follows:

$$\mathbf{Y_t} - \mathbf{Y_{t-1}} = \Delta\mathbf{Y_t} = \boldsymbol{\delta} + \left(\boldsymbol{\Theta_1} + \boldsymbol{\Theta}_2 - \mathbf{I}\right)\mathbf{Y_{t-1}} + \boldsymbol{\Theta_2}\Delta\mathbf{Y_{t-1}} + \mathbf{e_t} = \boldsymbol{\delta} + \boldsymbol{\Pi}\mathbf{Y_{t-1}} + \boldsymbol{\Gamma_1}\Delta\mathbf{Y_{t-1}} + \mathbf{e_t}$$

So from a VAR(2) you obtained a VEC(1) model:

If you find that a VAR(1) representation fits your data the best and wish to estimate a VEC(0) as follows:

$$\mathbf{Y_t} = \boldsymbol{\delta} + \boldsymbol{\Theta_1}\mathbf{Y_{t-1}} + \mathbf{e_t}$$

$$\mathbf{Y_t} - \mathbf{Y_{t-1}} = \Delta\mathbf{Y_t} = \boldsymbol{\delta} + \left(\boldsymbol{\Theta_1} - \mathbf{I}\right)\mathbf{Y_{t-1}} + \mathbf{e_t} = \boldsymbol{\delta} + \boldsymbol{\Pi}\mathbf{Y_{t-1}} + \mathbf{e_t}$$

Generally, any VAR(p) system can be rewritten as VEC(p-1).

$$\mathbf{Y_t} = \boldsymbol{\delta} + \sum_{i=1}^{p}\boldsymbol{\Theta}_i\mathbf{Y_{t-\mathit{i}}} + \mathbf{e_t}$$

$$\Delta\mathbf{Y_t} = \boldsymbol{\delta} + \left(\sum_{i=1}^{p}\boldsymbol{\Theta}_i - \mathbf{I}\right)\mathbf{Y_{t-1}} + \sum_{i=1}^{p-1}\left(-\sum_{j=i+1}^{p}\Theta_j\right)\Delta\mathbf{Y_{t-\mathit{i}}} + \mathbf{e_t} = \boldsymbol{\delta} + \boldsymbol{\Pi}\mathbf{Y_{t-1}} + \sum_{i=1}^{p-1}\boldsymbol{\Gamma}_i\Delta\mathbf{Y_{t-\mathit{i}}} + \mathbf{e_t}$$

Let us assume that we have a two-variate VAR(1) system:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}\begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}$$

The matrix **Π** from the VEC(0) representation is going to be:

$$\mathbf{\Pi} = \begin{pmatrix} \theta_{11}-1 & \theta_{12} \\ \theta_{21} & \theta_{22}-1 \end{pmatrix}$$

Obviously, if $\theta_{11}=1, \theta_{22}=1, \theta_{12}=\theta_{21}=0$ the above is a zero matrix, and this can be written as follows:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} \text{ or}$$

$y_{1t} = \delta_1 + y_{1t-1} + e_{1t}$ and $y_{2t} = \delta_2 + y_{2t-1} + e_{2t}$

That is, both y_1 and y_2 are random walk processes and independent of each other (no cointegration). In this case the 2x2 matrix **Π** is a zero matrix so its rank is by definition 0.

So, generally, when matrix Π has zero rank, we have non-stationary variables that are not cointegrated.

What if the rank of the matrix **Π** is less than k but higher than zero, so it has a reduced rank?
In this particular case it means that, if you have:

$$\mathbf{\Pi} = \begin{pmatrix} \theta_{11}-1 & \theta_{12} \\ \theta_{21} & \theta_{22}-1 \end{pmatrix}$$

then the trace should be zero (the matrix is singular):

$$(\theta_{11}-1)(\theta_{22}-1) - \theta_{21}\theta_{12} = 0$$

This is possible if $c(\theta_{11}-1) = \theta_{21}$ and $c\theta_{12} = \theta_{22}-1$, so the two rows of columns are linearly dependent.

Since $\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ $\mathbf{\Pi} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{21} \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{11} & \alpha_{11}\beta_{21} \\ \alpha_{21}\beta_{11} & \alpha_{21}\beta_{21} \end{pmatrix}$ obviously the two rows (columns) of the matrix **Π** are dependent. So you can create the 2x2 matrix as a product of a 2x1 and a 1x2 vectors. This is cointegration, and you have a single cointegrating vector. Of course it is possible that you normalize the element of the cointegrating vector for one of the variables to one. If you choose, say, β_{11}=1, then:

$$\mathbf{\Pi} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta_{21}}{\beta_{11}} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{11}\frac{\beta_{21}}{\beta_{11}} \\ \alpha_{21} & \alpha_{21}\frac{\beta_{21}}{\beta_{11}} \end{pmatrix}.$$

We can conclude that if the rank of Π is between zero and k, then your series are cointegrated and the number of cointegrating vectors equals the rank of Π.

Now we have only one case, when the matrix **Π** is of full rank.

$$\mathbf{\Pi} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{12} & \alpha_{11}\beta_{21} + \alpha_{12}\beta_{22} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{12} & \alpha_{21}\beta_{21} + \alpha_{22}\beta_{22} \end{pmatrix}$$

This means that you can only create your matrix **Π** as the product of two 2x2 matrices. In this case your series are stationary so they cannot be cointegrated by definition.
Why?

Let us look at the VEC again:

$$\Delta \mathbf{Y_t} = \boldsymbol{\delta} + \boldsymbol{\Pi} \mathbf{Y_{t-1}} + \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_i \Delta \mathbf{Y_{t-\mathit{i}}} + \mathbf{e_t}$$

The whole methodology is based on the assumption that your endogenous (dependent) variables are I(1). As such $\Delta \mathbf{Y_t}$ and any lags of it are stationary. We know that the residual is also stationary (this is a dynamic model, so $\boldsymbol{\Pi} \mathbf{Y_{t-1}}$ has to be stationary as well. If your y variables happen to be stationary without being cointegrated, then they had to be stationary already.

If you matrix Π is of full rank, your dependent variables are stationary and cannot be cointegrated.

10.c The Johansen test of cointegration

The main objective is that after you estimate the matrix Π, you determine its rank. In linear algebra courses you do this simply by some kind of elementary basis transformation so that you find out how many of the k rows of columns of the matrix are linearly independent. This would not be very useful now, so instead we test the rank of the kxk matrix **Π** using the eigenvalue approach.

What is the eigenvalue? Let us assume you have a kxk square matrix **A** and **c** is a kx1 vector. Now we say that **c** is the eigenvector of matrix **A** is there exist such a scalar λ that:

$$\mathbf{Ac} = \lambda \mathbf{c}$$

the scalar λ is called the eigenvalue of A.

Now, the main point here is, that **A** may have at most k eigenvalues. The number of non-zero eigenvalues equals the rank of matrix **A**.

So, without knowing how to calculate the eigenvalue (if you are interested, just look it up, you will find it quite familiar after the first lecture), you can just use some software to calculate it (you have some of them online, just google it).

Say you have a matrix like:

$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ its eigenvalues are 1 and 1, so both of them are non-zero: this matrix has the rank of two.

If you have a different matrix, like:

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2/3 & 1 \end{pmatrix}$ then the two eigenvalues are: 0 and 3, so the rank of the matrix is one. This means that one of them can be created as the linear combination of the other. And indeed, dividing row 1 by 3 gives the second row.

You have two tests. Both of them are based on an estimate of the eigenvalues of matrix **Π**, denoted as λ_i, i=1…k. The eigenvalues are ordered from the largest (i=1) to the smallest (i=k).

The trace test has the following statistics:

$\lambda_{trace}(r) = -T \ln(1-\lambda_r)$ where r is the rank of the matrix **Π** (number of cointegrating vectors) in the null-hypothesis. The null hypothesis is that λ_r is different from zero. In this case $\ln(1-\lambda_i)$ should have a negative value. The alternative hypothesis is that $\lambda_r = 0$ but then $\ln(1-\lambda_r) = 0$.

So the trace test has the null-hypothesis that the rank of **Π** is less than or equal to r.
An alternative test is ***the maximum eigenvalue test***. Here you have the null-hypothesis that the rank equals r, against the alternative that it is r+1. The test statistics is:

$$\lambda_{\max}(r, r+1) = -T \ln(1-\lambda_{r+1})$$

Let us put this into use!
I simulated two time series, y and x in a way that they are cointegrated. The first step is to have a proper VAR representation. Be careful, the test results may be very sensitive to your choice of the order of the VAR (an obvious weakness of this methodology) so you should always start with a standard VAR model!

VAR Lag Order Selection Criteria
Endogenous variables: Y X
Exogenous variables: C
Date: 03/08/12 Time: 19:27
Sample: 2 100
Included observations: 92

Lag	LogL	LR	FPE	AIC	SC	HQ
0	-252.8740	NA	0.873704	5.540738	5.595560	5.562865
1	-191.1067	119.5062	0.248881	4.284929	4.449393*	4.351308
2	-184.9513	11.64182	0.237528*	4.238071*	4.512179	4.348704*
3	-184.0929	1.586105	0.254413	4.306368	4.690118	4.461253
4	-182.9648	2.035608	0.270983	4.368799	4.862192	4.567937
5	-178.2620	8.280987	0.267153	4.353521	4.956558	4.596912
6	-171.0900	12.31710*	0.249728	4.284565	4.997244	4.572208
7	-168.4569	4.407547	0.257792	4.314281	5.136603	4.646177
8	-166.5506	3.108119	0.270531	4.359796	5.291761	4.735945

* indicates lag order selected by the criterion
LR: sequential modified LR test statistic (each test at 5% level)
FPE: Final prediction error
AIC: Akaike information criterion
SC: Schwarz information criterion
HQ: Hannan-Quinn information criterion

It seems that a VAR(2) is fine. Let us have a look at the residual diagnostics:

VAR Residual Serial Correlation LM T...
Null Hypothesis: no serial correlation ...
Date: 03/08/12 Time: 19:27
Sample: 2 100
Included observations: 98

Lags	LM-Stat	Prob
1	4.422875	0.3518
2	9.220740	0.0558
3	1.565077	0.8151
4	2.834276	0.5859
5	5.482298	0.2413
6	10.58876	0.0316
7	3.730927	0.4436
8	4.921002	0.2955
9	1.330849	0.8561
10	2.427246	0.6577
11	11.28086	0.0236
12	0.534112	0.9701

Probs from chi-square with 4 df.

VAR Residual Normality Tests
Orthogonalization: Cholesky (Lutkepohl)
Null Hypothesis: residuals are multivariate normal
Date: 03/08/12 Time: 19:28
Sample: 2 100
Included observations: 98

Component	Skewness	Chi-sq	df	Prob.
1	-0.251818	1.035730	1	0.3088
2	0.284935	1.326069	1	0.2495
Joint		2.361799	2	0.3070

Component	Kurtosis	Chi-sq	df	Prob.
1	2.692560	0.385953	1	0.5344
2	3.639687	1.670900	1	0.1961
Joint		2.056853	2	0.3576

Component	Jarque-Bera	df	Prob.
1	1.421684	2	0.4912
2	2.996969	2	0.2235
Joint	4.418653	4	0.3523

VAR Residual Heteroskedasticity Tests: No Cross Terms (only levels and squares)
Date: 03/08/12 Time: 19:28
Sample: 2 100
Included observations: 98

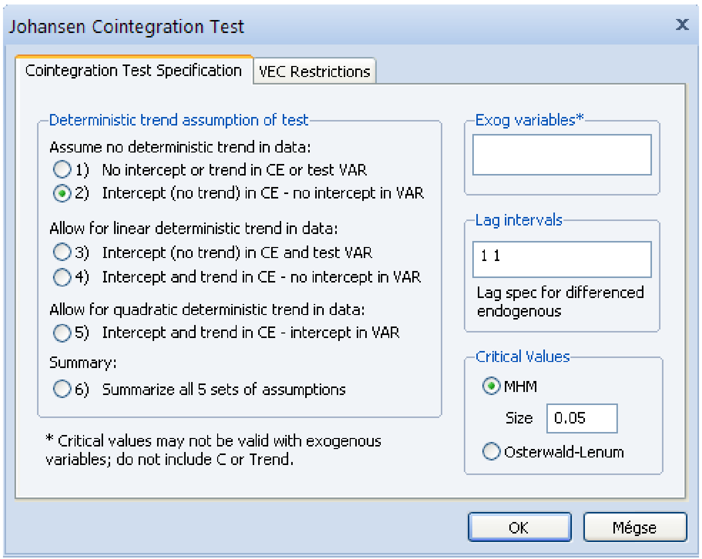
Joint test:

Chi-sq	df	Prob.
18.12339	24	0.7970

Individual components:

Dependent	R-squared	F(8,89)	Prob.	Chi-sq(8)	Prob.
res1*res1	0.067014	0.799081	0.6049	6.567379	0.5839
res2*res2	0.033204	0.382075	0.9276	3.253943	0.9174
res2*res1	0.071952	0.862529	0.5511	7.051315	0.5311

It seems all right (of course, I created the data in that way…), so we run a cointegration test:



Pay good attention to this menu: you may have several different assumption regarding the nature of the cointegrating relationship (now I chose number 2, because I created the relationship, so I know that number 2 is correct). Beware: you need to give the lag order of the VEC at the right-hand side (lag intervals) which is one less than the order of the VAR. We had a VAR(2) so I give here 1 1, meaning that we use a VEC(1) model. Now we have the output of the Johansen test.

Date: 03/08/12 Time: 19:32
Sample (adjusted): 3 100
Included observations: 98 after adjustments
Trend assumption: No deterministic trend (restricted constant)
Series: Y X
Lags interval (in first differences): 1 to 1

Unrestricted Cointegration Rank Test (Trace)

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	0.05 Critical Value	Prob.**
None *	0.383746	53.02539	20.26184	0.0000
At most 1	0.055386	5.583948	9.164546	0.2254

Trace test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)

Hypothesized No. of CE(s)	Eigenvalue	Max-Eigen Statistic	0.05 Critical Value	Prob.**
None *	0.383746	47.44145	15.89210	0.0000
At most 1	0.055386	5.583948	9.164546	0.2254

Max-eigenvalue test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

Unrestricted Cointegrating Coefficients (normalized by b'*S11*b=I):

Y	X	C
-0.063173	-2.724519	3.206938
0.408486	-0.701224	-2.166419

Unrestricted Adjustment Coefficients (alpha):

D(Y)	0.201534	-0.218415
D(X)	0.363488	0.035421

1 Cointegrating Equation(s): Log likelihood -203.2679

Normalized cointegrating coefficients (standard error in parentheses)

Y	X	C
1.000000	43.12804	-50.76455
	(5.77040)	(6.09718)

Adjustment coefficients (standard error in parentheses)

D(Y)	-0.012731
	(0.00624)
D(X)	-0.022963
	(0.00314)

This is the test output of the trace test: you can see that it clearly rejects the null-hypothesis that the rank is 0, while it cannot reject that the rank is 1.

This is the test output of the max eigenvalue test: you can see that it clearly rejects the null-hypothesis that the rank is 0 to the alternative that it is 1. If you have just two variables, the two tests are equivalent.

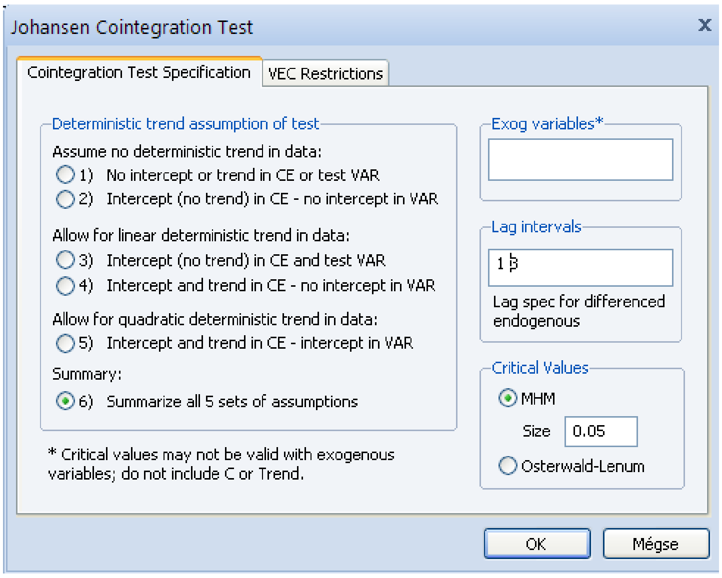
These are the non-normalized elements of the beta and the alpha matrix under no assumption regarding the number of cointegrating vectors.

These are the normalized elements of the beta and the alpha matrix under the assumption that you have one cointegrating vector. The Eviews will automatically normalize the coefficient of the first variable.

It is possible that you obtain conflicting results from the two tests. In this case, since its small-sample properties are better, you should rather prefer the results from the maximum eigenvalue test.

10.d. Real data examples

Let us return to tablef5.1 that we used for the VAR/SVAR exercise.
We have the basic model where we were interested in the relation between the log of real GDP and inflation rate. Let us look for evidence of a long-run relationship!
As first step we estimate the best VAR(p) system we can. Last time we agreed on a VAR(4).
Now we run a series of cointegration tests:



We tell the Eviews that we think in terms of a VEC(3) system. We do not know the exact form of the cointegrating vector however, so we ask the software to test a wide range of possibilities. Let us got through of these:

1) means that your cointegrating vector looks $\ln y_t = \beta_1 \text{infl}_t$, that is you force an assumption that with zero inflation you should have lny=0 (y=1) in the long-run, which is obviously false.
2) $\ln y_t = \beta_0 + \beta_1 \text{infl}_t$, now you allow that with zero inflation, the GDP can be different than one. In this case you assume that the ECM equation looks like: $\Delta \ln y_t = \gamma_1 \Delta \ln \text{infl}_t + \alpha_1 \left(\ln y_{t-1} - \beta_0 - \beta_1 \text{infl}_{t-1}\right) + e_{1t}$
3) In this case you allow for a constant in both part, so both in the cointegrating vector and outside of it.

   $\ln y_t = \beta_0 + \beta_1 \text{infl}_t$ and $\Delta \ln y_t = \delta_1 + \gamma_1 \Delta \ln \text{infl}_t + \alpha_1 \left(\ln y_{t-1} - \beta_0 - \beta_1 \text{infl}_{t-1}\right) + e_{1t}$

   If you think further about this model, you will find that by having a constant in your equation for the change of a dependent variable, you actually introduce a factor of constant change in it. So you assume that lny has some deterministic trend.
4) The same can be achieved by introducing a linear trend in the cointegrating vector while allowing for an intercept outside of it:

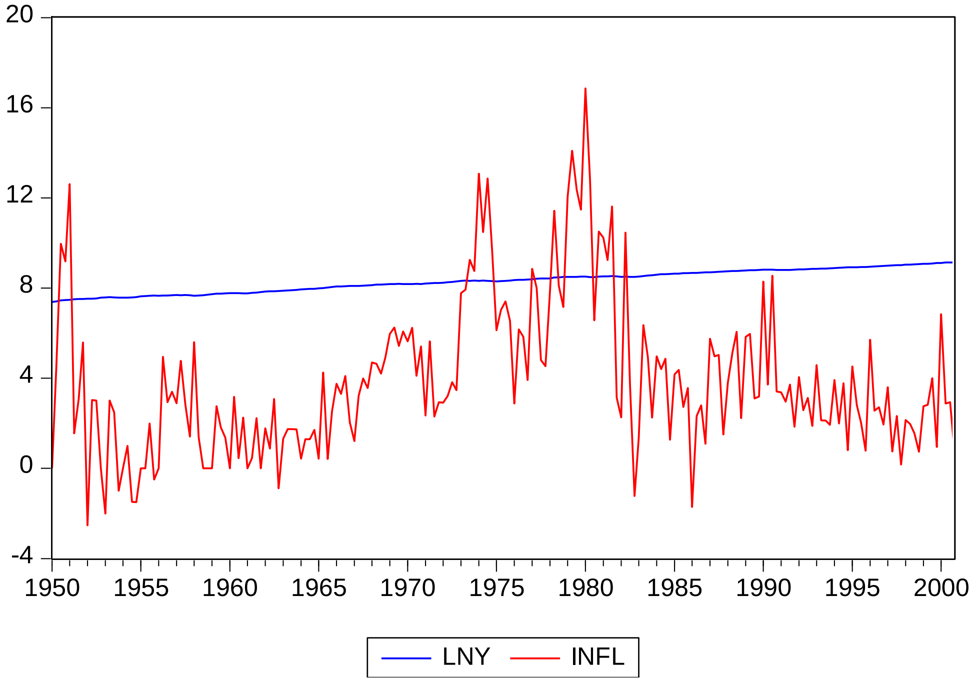
   $\ln y_t = \beta_0 + \beta_1 \text{infl}_t + \beta_2 t$ $\Delta \ln y_t = \delta_1 + \gamma_1 \Delta \ln \text{infl}_t + \alpha_1 \left(\ln y_{t-1} - \beta_0 - \beta_1 \text{infl}_{t-1} - \beta_2 t\right) + e_{1t}$

   The final result is the same: you will now have a deterministic trend in lny, but it will be because the long-run relationship is changing.
5) Finally, You can add a linear trend in- and outside of the cointegrating vector: $\ln y_t = \beta_0 + \beta_1 \text{infl}_t + \beta_2 t$

   $\Delta \ln y_t = \delta_{11} + \delta_{12} t + \gamma_1 \Delta \ln \text{infl}_t + \alpha_1 \left(\ln y_{t-1} - \beta_0 - \beta_1 \text{infl}_{t-1} - \beta_2 t\right) + e_{1t}$

   Since now you assume that the change of lny exhibits a linear trend, its level should have a quadratic trend.

Of course, you can look at your data for a hint:



It seems that at least the log of real GDP seem to have a clear trend, but still, only looking at the graphs will not give you the definitive answer on which assumption is correct.

When you have no idea which one is the correct one, you should test for all possibilities:

Date: 03/09/12 Time: 11:48
Sample: 1950Q1 2000Q4
Included observations: 200
Series: LNY INFL
Lags interval: 1 to 3

Selected (0.05 level*) Number of Cointegrating Relations by Model

Data Trend:	None	None	Linear	Linear	Quadratic
Test Type	No Intercept No Trend	Intercept No Trend	Intercept No Trend	Intercept Trend	Intercept Trend
Trace	2	2	0	1	2
Max-Eig	2	2	0	1	2

*Critical values based on MacKinnon-Haug-Michelis (1999)

Information Criteria by Rank and Model

Data Trend:	None	None	Linear	Linear	Quadratic
Rank or No. of CEs	No Intercept No Trend	Intercept No Trend	Intercept No Trend	Intercept Trend	Intercept Trend
	Log Likelihood by Rank (rows) and Model (columns)				
0	195.6982	195.6982	213.0787	213.0787	213.3295
1	214.3277	214.9559	219.2261	224.2201	224.3000
2	218.6317	219.6976	219.6976	229.1938	229.1938
	Akaike Information Criteria by Rank (rows) and Model (columns)				
0	-1.836982	-1.836982	-1.990787	-1.990787	-1.973295
1	-1.983277	-1.979559	-2.012261	-2.052201*	-2.043000
2	-1.986317	-1.976976	-1.976976	-2.051938	-2.051938
	Schwarz Criteria by Rank (rows) and Model (columns)				
0	-1.639083	-1.639083	-1.759905*	-1.759905*	-1.709430
1	-1.719412	-1.699202	-1.715412	-1.738861	-1.713168
2	-1.656485	-1.614161	-1.614161	-1.656140	-1.656140

Here we have several choices, but the information criteria seem to favor two options. Either that we have two non-stationary regressions that are not cointegrated (look for the stars in the Schwarz Information Criterion row with zero rank…) or that we have a single cointegrating vector and a linear trend in data (this is favored by the Akaike Information Criterion).

Let us carry out the cointegration test with a single cointegrating vector under assumption no. 4.

Date: 03/09/12 Time: 16:56
Sample (adjusted): 1951Q1 2000Q4
Included observations: 200 after adjustments
Trend assumption: Linear deterministic trend (restricted)
Series: LNY INFL
Lags interval (in first differences): 1 to 3

Unrestricted Cointegration Rank Test (Trace)

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	0.05 Critical Value	Prob.**
None *	0.105432	32.23023	25.87211	0.0070
At most 1	0.048520	9.947318	12.51798	0.1296

Trace test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)

Hypothesized No. of CE(s)	Eigenvalue	Max-Eigen Statistic	0.05 Critical Value	Prob.**
None *	0.105432	22.28292	19.38704	0.0184
At most 1	0.048520	9.947318	12.51798	0.1296

Max-eigenvalue test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

Unrestricted Cointegrating Coefficients (normalized by b'*S11*b=I):

LNY	INFL	@TREND(50Q2)
-29.25099	0.375714	0.235185
13.74678	0.227999	-0.112569

Unrestricted Adjustment Coefficients (alpha):

D(LNY)	-3.83E-05	-0.001990
D(INFL)	-0.724506	0.024566

1 Cointegrating Equation(s): Log likelihood 224.2201

Normalized cointegrating coefficients (standard error in parentheses)

LNY	INFL	@TREND(50Q2)
1.000000	-0.012844	-0.008040
	(0.00264)	(0.00013)

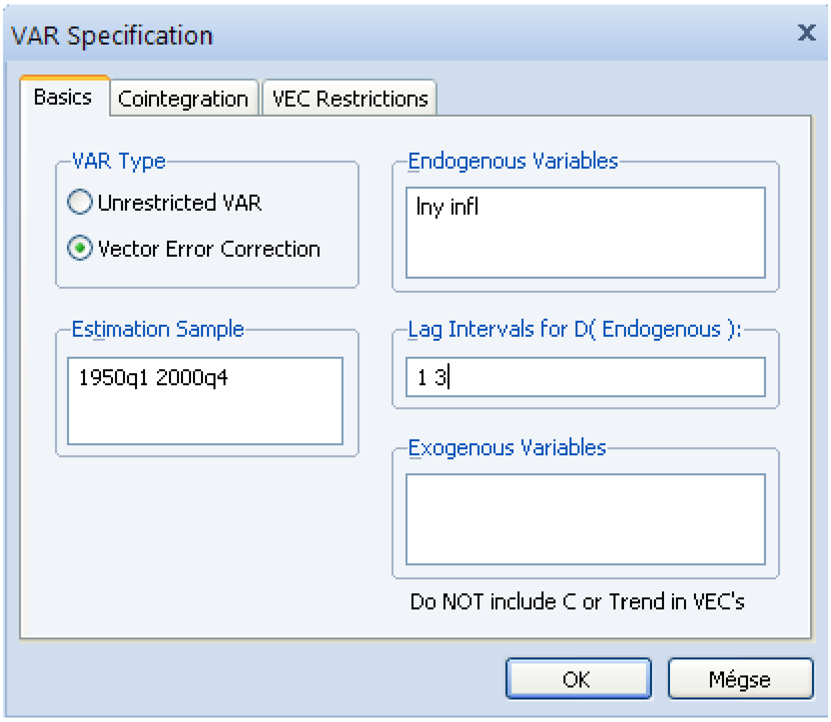
Adjustment coefficients (standard error in parentheses)

D(LNY)	0.001121
	(0.01908)
D(INFL)	21.19253
	(4.46126)

What we find now is that the cointegrating vector is: $\ln y_t = \beta_0 + 0.128 \cdot \text{infl}_t + 0.008 \cdot t$ the standard errors are much lower than the coefficients (respective t-stats would be 4.86 and 61.84 respectively) so we can be certain that the elements of the vector are significant at 1%.

The adjustment coefficients tell us a story of asymmetrical adjustment: The coefficient for lny is insignificant, meaning that real GDP does not seem to adapt to the equilibrium relation. On the other hand, the rate of inflation has a significant coefficient that shows adjustment from part of the inflation. So if inflation has a permanent shock the equilibrium value of the log of real GDP will also rise. But the real GDP will not react by adjusting to the new level; instead it is rather the inflation rate that will adjust and go down again.

The VEC system is estimated as follows:



Where you are going te get the following output:

We can also test for the presence of cointegration in the four equation VAR(6) system that included the natural log of real GDP, CPI, M1 and the level of treasury bill rate.
The cointegration test leads to the following:

Date: 03/10/12 Time: 13:49
Sample: 1950Q1 2000Q4
Included observations: 198
Series: LNY LNM LNCPI TBILRATE
Lags interval: 1 to 5

Selected (0.1 level*) Number of Cointegrating Relations by Model

Data Trend:	None	None	Linear	Linear	Quadratic
Test Type	No Intercept No Trend	Intercept No Trend	Intercept No Trend	Intercept Trend	Intercept Trend
Trace	3	4	2	1	2
Max-Eig	3	4	2	0	0

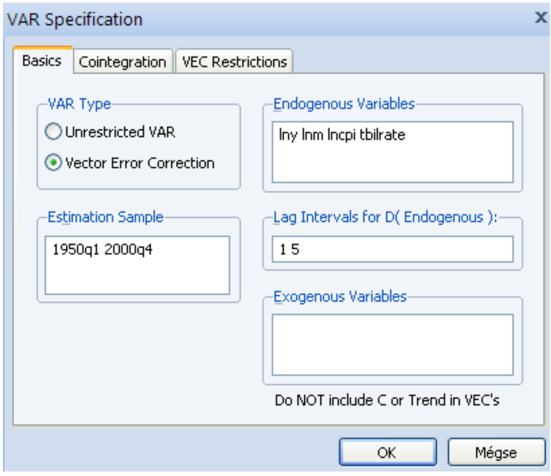
*Critical values based on MacKinnon-Haug-Michelis (1999)

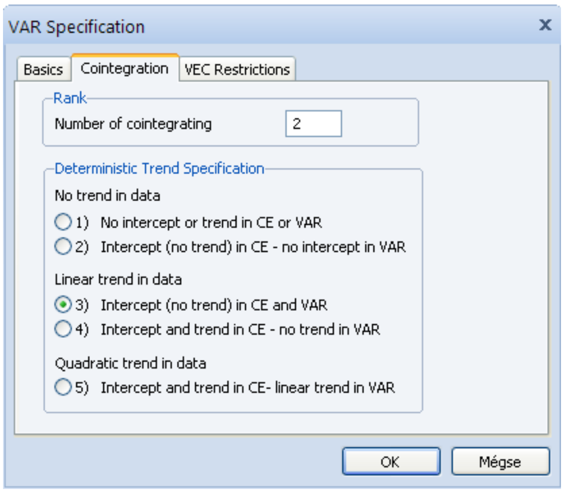
Information Criteria by Rank and Model

Data Trend:	None	None	Linear	Linear	Quadratic
Rank or No. of CEs	No Intercept No Trend	Intercept No Trend	Intercept No Trend	Intercept Trend	Intercept Trend
	Log Likelihood by Rank (rows) and Model (columns)				
0	1968.537	1968.537	1986.318	1986.318	1988.598
1	1990.694	1992.401	1999.924	2000.771	2002.504
2	1999.830	2002.564	2009.430	2011.238	2012.507
3	2004.879	2009.832	2014.602	2016.420	2016.885
4	2005.696	2014.651	2014.651	2020.467	2020.467
	Akaike Information Criteria by Rank (rows) and Model (columns)				
0	-19.07613	-19.07613	-19.21533	-19.21533	-19.19796
1	-19.21913	-19.22627	-19.27196	-19.27042	-19.25762
2	-19.23061	-19.23802	-19.28718*	-19.28524	-19.27785
3	-19.20080	-19.22053	-19.25860	-19.24666	-19.24127
4	-19.12824	-19.17830	-19.17830	-19.19663	-19.19663
	Schwarz Criteria by Rank (rows) and Model (columns)				
0	-17.74753	-17.74753	-17.82031*	-17.82031*	-17.73651
1	-17.75768	-17.74821	-17.74408	-17.72593	-17.66331
2	-17.63630	-17.61050	-17.62643	-17.59128	-17.55067
3	-17.47363	-17.44354	-17.46500	-17.40324	-17.38124
4	-17.26821	-17.25184	-17.25184	-17.20374	-17.20374

If we again prefer the AIC, we come at the conclusion that there are two cointegrating vectors and some linear trend in the data.

We can estimate the basic VEC(5) as follows:





The estimates of the beta and alpha vectors are displayed as follows:

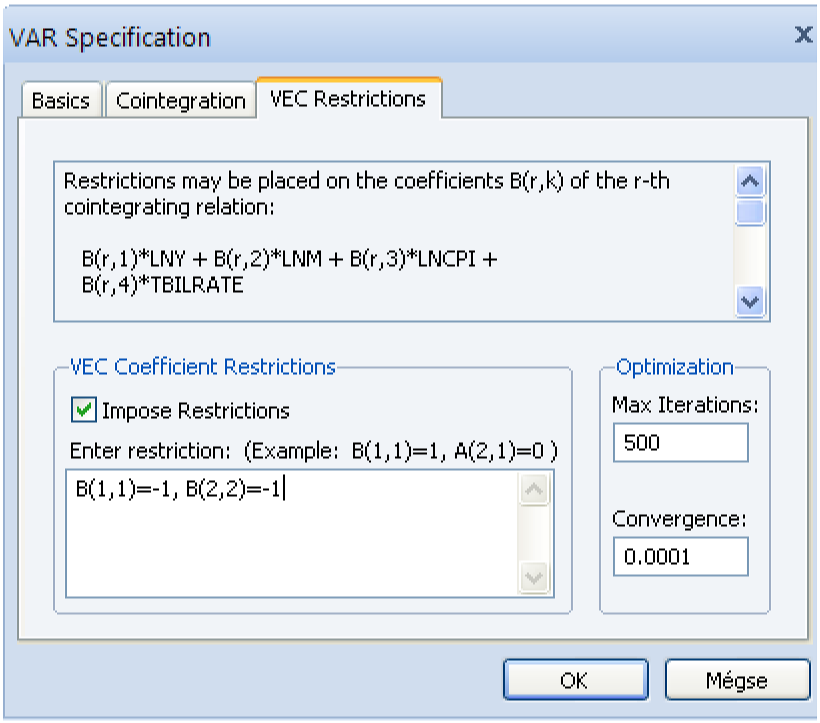
Vector Error Correction Estimates
Date: 03/10/12 Time: 13:54
Sample (adjusted): 1951Q3 2000Q4
Included observations: 198 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegrating Eq:	CointEq1	CointEq2		
LNY(-1)	1.000000	0.000000		
LNM(-1)	0.000000	1.000000		
LNCPI(-1)	-0.730356	-1.275770		
	(0.05350)	(0.04298)		
	[-13.6515]	[-29.6800]		
TBILRATE(-1)	0.064127	0.088703		
	(0.01859)	(0.01494)		
	[3.44899]	[5.93793]		
C	-4.867703	0.353406		
Error Correction:	**D(LNY)**	**D(LNM)**	**D(LNCPI)**	**D(TBILRATE)**
CointEq1	0.023786	0.026058	0.010165	0.612010
	(0.00915)	(0.00854)	(0.00512)	(0.63263)
	[2.60045]	[3.04969]	[1.98673]	[0.96741]
CointEq2	-0.035076	-0.027939	-0.003366	-1.961488
	(0.01193)	(0.01115)	(0.00668)	(0.82539)
	[-2.93920]	[-2.50612]	[-0.50421]	[-2.37645]
D(LNY(-1))	0.184981	-0.105055	-0.014981	13.88851
	(0.08384)	(0.07832)	(0.04690)	(5.79847)
	[2.20642]	[-1.34139]	[-0.31945]	[2.39520]
D(LNY(-2))	-0.017286	-0.030159	-0.079734	13.11621
	(0.08312)	(0.07765)	(0.04650)	(5.74905)
	[-0.20795]	[-0.38839]	[-1.71482]	[2.28146]

You can see that there are four restrictions already in the cointegrating vector $\beta_{11}=\beta_{22}=1$, that is lnY and lnM1 are normalized to one respectively. The rest of the restrictions are introduced so that the two vectors are surely independent of each other $\beta_{12}=\beta_{21}=0$, but this is not necessarily true. The alpha coefficients are also reported, and some of them are seemingly not significant: It seems, for example, that the log CPI does not adjust to its respective cointegrating relation.

What we can do is to change the restrictions. Of course there is some limitation here: in order to have standard errors and a specification test you need to keep the cointegrating vector identified.

You can introduce restrictions by reestiamting the VEC:



Here you can assign values to the elements of both the cointegrating vectors and the adjustment vectors. Now, for example, I introduced only two restrictions (obviously, this will render our vectors unidentified). I choose now, that I normalize the elements of the two cointegrating vectors to minus one for the lnY and the lnM. by using minus one instead of positive one, we gain only one thing: you need not to multiply the coefficient by minus one to have their right sign.

Vector Error Correction Estimates
Date: 03/10/12 Time: 14:05
Sample (adjusted): 1951Q3 2000Q4
Included observations: 198 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegration Restrictions:
B(1,1)=-1, B(2,2)=-1
Convergence achieved after 1 iterations.
Not all cointegrating vectors are identified
Restrictions are not binding (LR test not available)

Cointegrating Eq:	CointEq1	CointEq2
LNY(-1)	-1.000000	0.609714
LNM(-1)	0.355709	-1.000000
LNCPI(-1)	0.276554	0.830462
TBILRATE(-1)	-0.032575	-0.049604
C	4.993413	-3.321311

Error Correction:	D(LNY)	D(LNM)	D(LNCPI)	D(TBILRATE)
CointEq1	-0.003064	-0.011523	-0.010360	0.745654
	(0.00548)	(0.00512)	(0.00306)	(0.37875)
	[-0.55953]	[-2.25249]	[-3.38194]	[1.96872]
CointEq2	0.033986	0.023840	-0.000319	2.226724
	(0.01171)	(0.01094)	(0.00655)	(0.80962)
	[2.90335]	[2.18011]	[-0.04874]	[2.75035]

We can introduce additional restrictions based on theory: we know for example, that in the long-run, if money supply increases by one percentage, CPI should also go up by one percentage. This would then introduce $\beta_{23}=1$ as additional restriction. We can also argue that changes in prices should not have any long-run effect on the log of real GDP that leaves us with a second restriction: $\beta_{13}=0$.

Vector Error Correction Estimates
Date: 03/10/12 Time: 14:09
Sample (adjusted): 1951Q3 2000Q4
Included observations: 198 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegration Restrictions:
B(1,1)=-1, B(2,2)=-1, B(1,3)=0, B(2,3)=1
Convergence achieved after 1 iterations.
Restrictions identify all cointegrating vectors
Restrictions are not binding (LR test not available)

Cointegrating Eq:	CointEq1	CointEq2
LNY(-1)	-1.000000	0.377583
		(0.04043)
		[9.33908]
LNM(-1)	0.572483	-1.000000
	(0.02843)	
	[20.1376]	
LNCPI(-1)	0.000000	1.000000
TBILRATE(-1)	-0.013346	-0.064490
	(0.01176)	(0.00966)
	[-1.13522]	[-6.67650]
C	5.070022	-2.191366

Error Correction:	D(LNY)	D(LNM)	D(LNCPI)	D(TBILRATE)
CointEq1	-0.013449	-0.019786	-0.011347	0.164082
	(0.00708)	(0.00661)	(0.00396)	(0.48937)
	[-1.90075]	[-2.99350]	[-2.86691]	[0.33529]
CointEq2	0.027377	0.016611	-0.003130	2.055423
	(0.00979)	(0.00914)	(0.00547)	(0.67683)
	[2.79755]	[1.81710]	[-0.57180]	[3.03683]
D(LNY(-1))	0.184981	-0.105055	-0.014981	13.88851

You can also introduce restrictions regarding the adjustment parameters:
as we observed before, the log of CPI does not seem to adjust to the equilibrium value of log M1, so it can be set to zero: α_{32}=0, additionally we find the same for treasury bill rate and the equilibrium value of log real GDP: α_{41}=0.

Vector Error Correction Estimates
Date: 03/10/12 Time: 14:13
Sample (adjusted): 1951Q3 2000Q4
Included observations: 198 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegration Restrictions:
B(1,1)=-1, B(2,2)=-1, B(1,3)=0, B(2,3)=1,
A(3,2)=0, A(4,1)=0
Convergence achieved after 20 iterations.
Restrictions identify all cointegrating vectors
LR test for binding restrictions (rank = 2):
Chi-square(2) 0.221446
Probability 0.895187

Cointegrating Eq:	CointEq1	CointEq2		
LNY(-1)	-1.000000	0.390453		
		(0.04582)		
		[8.52093]		
LNM(-1)	0.573217	-1.000000		
	(0.02954)			
	[19.4065]			
LNCPI(-1)	0.000000	1.000000		
TBILRATE(-1)	-0.017036	-0.071576		
	(0.01223)	(0.01096)		
	[-1.39236]	[-6.52817]		
C	5.085396	-2.260835		
Error Correction:	**D(LNY)**	**D(LNM)**	**D(LNCPI)**	**D(TBILRATE)**
CointEq1	-0.014125	-0.019216	-0.011807	0.000000
	(0.00668)	(0.00642)	(0.00346)	(0.00000)
	[-2.11334]	[-2.99284]	[-3.41606]	[NA]
CointEq2	0.025154	0.014406	0.000000	2.007536
	(0.00880)	(0.00828)	(0.00000)	(0.53938)
	[2.85949]	[1.74020]	[NA]	[3.72191]

The specification test (for which you need to overidentify the cointegrating vector) does not reject the null hypothesis that this restricted form is at least as good as the basic case.
You can of course go further and test other theoretical restrictions.