

Advanced time-series analysis (University of Lund, Economic History Department)

30 Jan-3 February and 26-30 March 2012

Lecture 7 Conditional heteroscedasticity models: ARCH and GARCH techniques and their applications.

7.a Motivation and the basic ARCH model

Standard time series models are concerned about estimating the conditional expected value

of a process: $y_t = E(y_t | \mathbf{X}_t) + u_t = \beta_0 + \sum_{i=1}^p \beta_i x_{it} + u_t$

Where traditionally it is assumed that

$u_t \sim N(0, \sigma_u^2)$, or at least that the residuals are homoscedastic (σ_u^2 is constant and does not depend on \mathbf{X} or t) and serially uncorrelated (meaning that all information that can be used for prediction is incorporated in the model). I will refer to this equation as "level equation".

If the homoscedasticity condition was violated, we either had the standard errors adjusted (heteroscedasticity robust standard errors) or we used a Weighted Least Squares. This was done in order to have more efficient estimates of the level equation.

But the variance of the residual (or residual volatility) can also be of importance to us: it can be a measure of unpredictability or risk. While it is the most popular in financial econometrics, it has also been used in macroeconometrics (Engle 1982)¹ and economic history (Földvári and Van Leeuwen 2011)². Also it is a very often observed fact that there are more and less volatile periods, that is, volatility may be clustered (in short: volatility changes over time). This needs to be modeled.

The idea is the following:

The variance of the error term in the level equation can be written as:

$$\sigma_{u_t}^2 = \text{var}(u_t | u_{t-1}, u_{t-2}, \dots) = E[(u_t - E(u_t))^2 | u_{t-1}, u_{t-2}, \dots]$$

where we already assume that the variance may change over time (that is why the subscript t is added). So what we assume is that the variance of the residual can be explained by the past variances. Since $E(u_t) = 0$, the above equation can be rewritten purely in terms of the squared residuals:

$$\sigma_{u_t}^2 = E[u_t^2 | u_{t-1}, u_{t-2}, \dots] \text{ so basically: } \sigma_{u_t}^2 = \alpha_0 + \sum_{j=1}^m \alpha_j u_{t-j}^2 \text{ which is an Autoregressive}$$

Conditional Heteroscedasticity (ARCH(m)) model. If $\alpha_j > 0$ you have a clustering and also it is quite obvious that it will lead to a more leptokurtic (positive excess kurtosis) residuals from the level equation. Also, since residual variance is strictly positive, you have an assumption that the estimated (conditional) volatility is also positive.

The conditional volatility model is actually an estimation of the two equations:

$$\text{the level equation: } y_t = \beta_0 + \sum_{i=1}^p \beta_i x_{it} + u_t \text{ with } u_t \sim N(0, \sigma_{u_t}^2)$$

¹ Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50, 4: 987–1007.

² Földvári, P. and Van Leeuwen, B. (2011) Conditional heteroscedasticity in historical commodity price series *Cliometrica*, 5(2):165-186).

and the variance equation: $\sigma_{u_t}^2 = \alpha_0 + \sum_{j=1}^m \alpha_j u_{t-j}^2$

Very often the residual variance $\sigma_{u_t}^2$ is denoted by h_t .

Often the residual term from the level equation is written as:

$$u_t = \varepsilon_t \sigma_t = \varepsilon_t \sqrt{h_t} \quad \varepsilon_t \sim N(0,1)$$

where $\sigma_{u_t}^2 = h_t = \alpha_0 + \sum_{j=1}^m \alpha_j u_{t-j}^2$

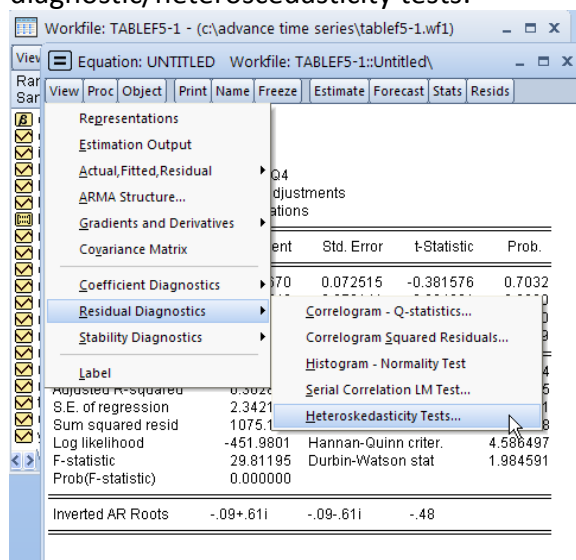
Of course, you will use the squared residuals to obtain some knowledge regarding the volatility of the series that are not the same. So: $\sigma_{u_t}^2 = u_t^2 + e_t$, $e_t \sim WN(0, \sigma_e^2)$. Basically you say here that your squared residual can be modeled by an AR(m) model:

$u_t^2 = \alpha_0 + \sum_{j=1}^m \alpha_j u_{t-j}^2 + e_t$, where $e_t = u_t^2 - \sigma_{u_t}^2 \sim WN(0, \sigma_e^2)$ Of course, if your series is

stationary, its variance should also be finite, so the variance equation should have the coefficient accordingly (it should be a stable process). If, for example, $\sigma_{u_t}^2 = \alpha_0 + u_{t-1}^2$, you will end up with a non-finite variance.

7.b Testing for ARCH type heteroscedasticity

The fundamental test for the need of a conditional heteroscedasticity model is one of the heteroscedasticity models and you can find it accordingly among the residual diagnostic/heteroscedasticity tests:



This requires however that you first have a model for the level equation.

If you have no structural model in mind you can always return to the Box-Jenkins methodology.

First of all, make the best possible test using the Box-Jenkins method. For this example I use the Table f5.1 and look for an ARIMA type model for the inflation.

Since unit-root tests suggest that the inflation is I(1) I take its first-difference and go on with the specification. Finally it turns out that an AR(2) model fits the change of inflation fine so we choose a ARIMA(2,1,0) model for it:

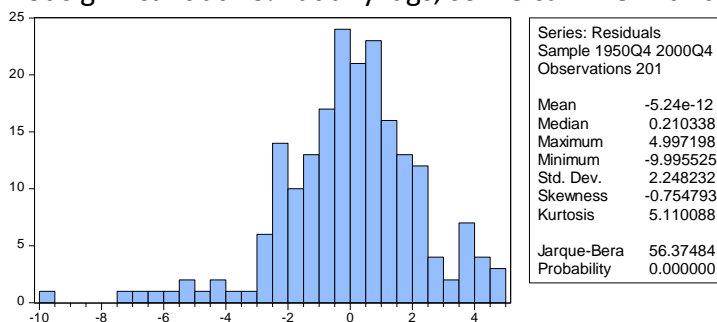
Dependent Variable: D(INFL)
 Method: Least Squares
 Date: 03/21/12 Time: 19:52
 Sample (adjusted): 1950Q4 2000Q4
 Included observations: 201 after adjustments
 Convergence achieved after 3 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.217108	0.345072	3.527111	0.0005
@QUARTER=2	-0.910511	0.566583	-1.607023	0.1097
@QUARTER=3	-1.652897	0.503548	-3.282504	0.0012
@QUARTER=4	-2.391285	0.563485	-4.243744	0.0000
AR(1)	-0.568147	0.066450	-8.549951	0.0000
AR(2)	-0.298428	0.066058	-4.517689	0.0000
R-squared	0.354596	Mean dependent var	-0.046490	
Adjusted R-squared	0.338048	S.D. dependent var	2.798501	
S.E. of regression	2.276873	Akaike info criterion	4.512880	
Sum squared resid	1010.910	Schwarz criterion	4.611486	
Log likelihood	-447.5444	Hannan-Quinn criter.	4.552780	
F-statistic	21.42731	Durbin-Watson stat	2.095259	
Prob(F-statistic)	0.000000			
Inverted AR Roots	-.28+.47i	-.28-.47i		

Date: 03/21/12 Time: 19:53
 Sample: 1950Q4 2000Q4
 Included observations: 201
 Q-statistic probabilities adjusted for 2 ARMA term(s)

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1	-0.062	-0.062	0.7904
		2	-0.069	-0.073	1.7548
		3	0.020	0.011	1.8411
		4	0.115	0.113	4.5639
		5	-0.084	-0.068	6.0211
		6	0.015	0.020	6.0653
		7	-0.038	-0.051	6.3655
		8	-0.106	-0.122	8.7304
		9	-0.083	-0.090	10.187
		10	-0.017	-0.054	10.252

Note that I found some seasonality in the series so I choose the simplest approach to correcting for this and included dummies for the quarters. The residual autocorrelation is not significant at 10% at any lags, so we can live with this.



The residual is not normally distributed though, that may be because of some unmodelled structure of the volatility. The ARCH test is carried out with 4 lags:

Heteroskedasticity Test: ARCH

F-statistic	3.824008	Prob. F(4,192)	0.0051
Obs*R-squared	14.53630	Prob. Chi-Square(4)	0.0058

Test Equation:

Dependent Variable: RESID^2
 Method: Least Squares
 Date: 03/21/12 Time: 19:54
 Sample (adjusted): 1951Q4 2000Q4
 Included observations: 197 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	2.900535	0.684111	4.239861	0.0000
RESID^2(-1)	0.032477	0.071735	0.452732	0.6513
RESID^2(-2)	0.020561	0.052785	0.389533	0.6973
RESID^2(-3)	0.157193	0.052449	2.997091	0.0031
RESID^2(-4)	0.082233	0.053538	1.535982	0.1262
R-squared	0.073788	Mean dependent var	4.333514	
Adjusted R-squared	0.054492	S.D. dependent var	7.542081	
S.E. of regression	7.333710	Akaike info criterion	6.847893	
Sum squared resid	10326.39	Schwarz criterion	6.931223	
Log likelihood	-669.5175	Hannan-Quinn criter.	6.881626	
F-statistic	3.824008	Durbin-Watson stat	2.010262	
Prob(F-statistic)	0.005150			

The level equation is:

$$\Delta \ln f_t = \beta_0 + \beta_1 \Delta \ln f_{t-1} + \beta_2 \Delta \ln f_{t-2} + \beta_4 D_t^{Q2} + \beta_5 D_t^{Q3} + \beta_6 D_t^{Q4} + \hat{u}_t$$

The test equation for the ARCH test is:

$$\hat{u}_t^2 = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{u}_{t-1}^2 + \hat{\alpha}_2 \hat{u}_{t-2}^2 + \hat{\alpha}_3 \hat{u}_{t-3}^2 + \hat{\alpha}_4 \hat{u}_{t-4}^2 + e_t$$

Where in case of no ARCH type structure in the residual volatility, you should expect that $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_3 = \hat{\alpha}_4 = 0$ which is the null hypothesis of the test. So basically you carry out a joint significance test on the test equation.

In the above case we find that the null hypothesis is rejected at 1%, so some further analysis may be warranted.

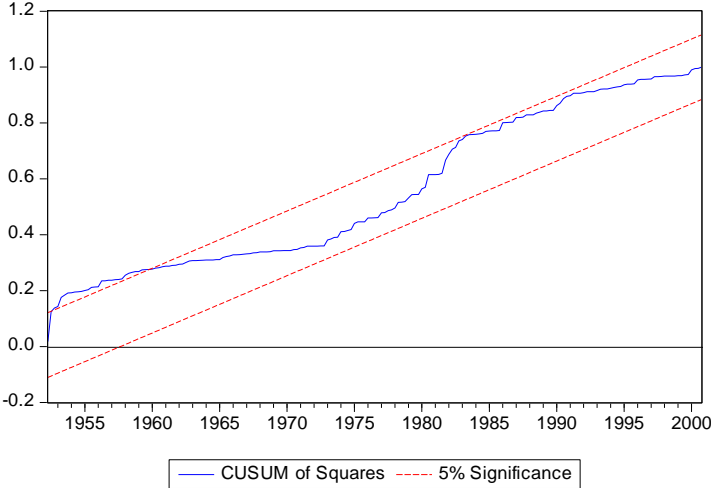
Alternatively, you can use the Q-test of the squared residuals:

Date: 03/26/12 Time: 10:42
 Sample: 1950Q4 2000Q4
 Included observations: 201

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
		1	0.162	0.162	5.3603	0.021
		2	0.157	0.134	10.410	0.005
		3	0.182	0.145	17.256	0.001
		4	0.116	0.057	20.067	0.000
		5	0.116	0.056	22.864	0.000
		6	0.035	-0.032	23.124	0.001
		7	0.076	0.033	24.353	0.001
		8	0.009	-0.037	24.372	0.002
		9	0.070	0.054	25.421	0.003
		10	-0.020	-0.056	25.508	0.004
		11	0.024	0.020	25.630	0.007
		12	0.008	-0.012	25.644	0.012
		13	-0.007	-0.003	25.653	0.019
		14	-0.009	-0.021	25.672	0.028
		15	-0.034	-0.023	25.920	0.039
		16	0.042	0.047	26.200	0.050

This shows us that the squared residuals from the regression show a degree of autocorrelation.

Finally you can rely on CUSUMSQ test as well, but this time you check for a structural instability of the squared recursive residuals, so it can show you if residual variance depends on time:



Again this suggests residual variance being dependent on time.

7.c. Estimating an ARCH model

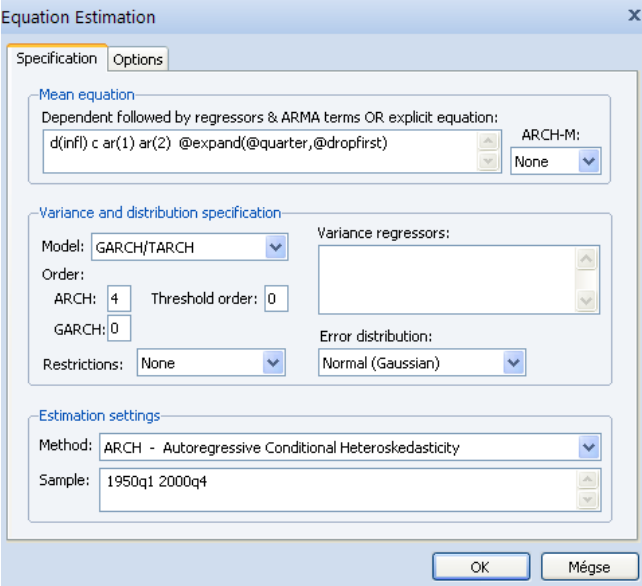
There are two options to estimate an ARCH model. The first is to estimate the level and variance equations as a system (that is simultaneously). This can be done with a Maximum Likelihood method. Another option is to do the estimation in two steps. First you estimate the level equation with an OLS/WLS or ML (in case of an ARMA type model ML is preferable).

If you use a least-squares approach, it may be worthwhile to use a WLS because of the heteroscedasticity of the residual.

After this, you can use a standard regression technique again on the squared residuals from the level equation, which is basically the same as the ARCH test. When will you go for a two-step approach? Especially in historical analysis you often have a lot of missing observations in your sample. In standard softwares, like Eviews, ML algorithms are not capable of treating missing observations, so you will be forced to go for an alternative.

If you use the built-in ARCH estimation procedure of the Eviews you should specify both equations:

Now we tell the EViews that the level equation should be a ARIMA(3,1,0) type model, while the variance equation should be a ARCH(4) type. We also say that we believe that the errors follow a normal distribution. We obtain the following results:



Seemingly the ARCH(4) was not the best idea as we have two insignificant coefficients.

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/21/12 Time: 19:57
 Sample (adjusted): 1950Q4 2000Q4
 Included observations: 201 after adjustments
 Convergence achieved after 64 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(7) + C(8)*RESID(-1)^2 + C(9)*RESID(-2)^2 + C(10)*RESID(-3)^2 + C(11)*RESID(-4)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	1.132906	0.294009	3.853300	0.0001
@QUARTER=2	-0.724855	0.526412	-1.376973	0.1685
@QUARTER=3	-1.541350	0.385132	-4.002136	0.0001
@QUARTER=4	-2.361262	0.619934	-3.808892	0.0001
AR(1)	-0.645461	0.072588	-8.892127	0.0000
AR(2)	-0.220693	0.076888	-2.870315	0.0041

Variance Equation				
C	2.372884	0.430868	5.507218	0.0000
RESID(-1)^2	0.104468	0.081250	1.285768	0.1985
RESID(-2)^2	0.203012	0.086632	2.343387	0.0191
RESID(-3)^2	-0.012946	0.053154	-0.243558	0.8076
RESID(-4)^2	0.201220	0.112670	1.785929	0.0741

R-squared	0.338391	Mean dependent var	-0.046490
Adjusted R-squared	0.321427	S.D. dependent var	2.798501
S.E. of regression	2.305281	Akaike info criterion	4.401536
Sum squared resid	1036.292	Schwarz criterion	4.582313
Log likelihood	-431.3543	Hannan-Quinn criter.	4.474686
Durbin-Watson stat	1.963664		

Inverted AR Roots -0.32+0.34i -0.32-0.34i

We can start to improve the model, and look at the information criteria for a lead in selection.

You can for example assume that there is seasonality in the variance as well, meaning that in some quarters you tend to have higher or lower variance of the residual.

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/21/12 Time: 20:07
 Sample (adjusted): 1950Q4 2000Q4
 Included observations: 201 after adjustments
 Convergence achieved after 38 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(7) + C(8)*RESID(-1)^2 + C(9)*RESID(-2)^2 + C(10)*RESID(-3)^2 + C(11)*RESID(-4)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.990775	0.389353	2.544670	0.0109
@QUARTER=2	-0.512658	0.691535	-0.741334	0.4585
@QUARTER=3	-1.443367	0.365718	-3.946663	0.0001
@QUARTER=4	-2.107415	0.591905	-3.560395	0.0004
AR(1)	-0.655580	0.062571	-10.47700	0.0000
AR(2)	-0.142266	0.067871	-2.096108	0.0361

Variance Equation				
C	5.675703	1.761057	3.222895	0.0013
RESID(-1)^2	0.059828	0.087319	0.685170	0.4932
RESID(-2)^2	0.349835	0.102478	3.413749	0.0006
RESID(-3)^2	-0.000705	0.063967	-0.011014	0.9912
RESID(-4)^2	0.057668	0.085258	0.676396	0.4988
@QUARTER=2	-3.439228	1.815070	-1.894819	0.0581
@QUARTER=3	-4.502706	1.812494	-2.484260	0.0130
@QUARTER=4	-4.356478	1.834437	-2.374831	0.0176

R-squared	0.313311	Mean dependent var	-0.046490
Adjusted R-squared	0.295704	S.D. dependent var	2.798501
S.E. of regression	2.348568	Akaike info criterion	4.356437
Sum squared resid	1075.576	Schwarz criterion	4.586518
Log likelihood	-423.8219	Hannan-Quinn criter.	4.449538
Durbin-Watson stat	1.982553		

Inverted AR Roots -0.33-0.19i -0.33+0.19i

We find that in the 2nd, 3rd and 4th quarters the volatility is lower than in the first. This is not very surprising, since most price changes occur in the beginning of the year.

You can also add or remove lags. In this case we find that an ARCH(5) model has the lowest AIC and SBC statistics.

Dependent Variable: D(INFL)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/21/12 Time: 21:17
Sample (adjusted): 1950Q4 2000Q4
Included observations: 201 after adjustments
Failure to improve Likelihood after 44 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(7) + C(8)*RESID(-1)^2 + C(9)*RESID(-2)^2 + C(10)*RESID(-3)^2 + C(11)*RESID(-4)^2 + C(12)*RESID(-5)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.772291	0.375487	2.056775	0.0397
@QUARTER=2	-0.440661	0.680544	-0.647514	0.5173
@QUARTER=3	-1.183635	0.381422	-3.103218	0.0019
@QUARTER=4	-1.732877	0.513043	-3.377647	0.0007
AR(1)	-0.669078	0.036979	-18.09362	0.0000
AR(2)	-0.171599	0.047686	-3.598536	0.0003

Variance Equation				
C	5.875675	1.779126	3.302562	0.0010
RESID(-1)^2	-0.009886	0.056182	-0.175610	0.8606
RESID(-2)^2	0.230926	0.110896	2.082378	0.0373
RESID(-3)^2	-0.026323	0.049885	-0.527661	0.5977
RESID(-4)^2	0.096752	0.091672	1.055415	0.2912
RESID(-5)^2	0.168250	0.084971	1.980084	0.0477
@QUARTER=2	-3.977172	1.793031	-2.218128	0.0265
@QUARTER=3	-4.570041	1.858094	-2.459532	0.0139
@QUARTER=4	-5.127838	1.772449	-2.893080	0.0038

R-squared	0.314309	Mean dependent var	-0.046490
Adjusted R-squared	0.296727	S.D. dependent var	2.798501
S.E. of regression	2.346861	Akaike info criterion	4.289331
Sum squared resid	1074.013	Schwarz criterion	4.535846
Log likelihood	-416.0777	Hannan-Quinn criter.	4.389081
Durbin-Watson stat	1.951206		

Inverted AR Roots	-.33-.24i	-.33+.24i
-------------------	-----------	-----------

The residual has no serial correlation significant at 1%:

Date: 03/21/12 Time: 21:17
Sample: 1950Q4 2000Q4
Included observations: 201
Q-statistic probabilities adjusted for 2 ARMA term(s)

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1 -0.022	-0.022	0.0960	
		2 -0.102	-0.103	2.2424	
		3 0.072	0.069	3.3244	0.068
		4 0.126	0.121	6.6294	0.036
		5 -0.085	-0.068	8.1464	0.043
		6 0.023	0.040	8.2586	0.083
		7 -0.034	-0.066	8.5024	0.131
		8 -0.106	-0.111	10.884	0.092
		9 -0.113	-0.116	13.584	0.059
		10 -0.045	-0.082	14.016	0.081
		11 -0.024	-0.019	14.141	0.117
		12 0.046	0.073	14.607	0.147

Some are significant at 5% though. You can add a further lag of the dependent variable to the level equation.

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/21/12 Time: 21:20
 Sample (adjusted): 1951Q1 2000Q4
 Included observations: 200 after adjustments
 Failure to improve Likelihood after 20 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(8) + C(9)*RESID(-1)^2 + C(10)*RESID(-2)^2 + C(11)*RESID(-3)^2 + C(12)*RESID(-4)^2 + C(13)*RESID(-5)^2

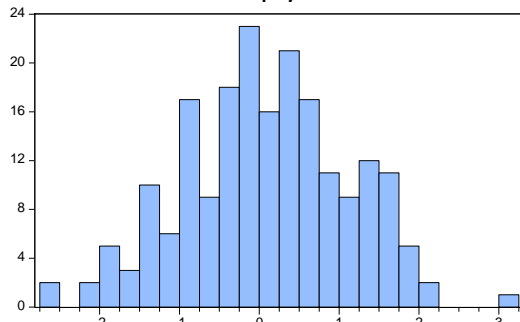
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.927012	0.450562	2.057458	0.0396
@QUARTER=2	-0.693129	0.868333	-0.798229	0.4247
@QUARTER=3	-1.336473	0.471617	-2.833812	0.0046
@QUARTER=4	-1.975558	0.630679	-3.132429	0.0017
AR(1)	-0.739872	0.064948	-11.39170	0.0000
AR(2)	-0.365289	0.085480	-4.273412	0.0000
AR(3)	-0.213277	0.068433	-3.116566	0.0018

Variance Equation				
C	5.584444	1.481224	3.770154	0.0002
RESID(-1)^2	-0.014776	0.034152	-0.432650	0.6653
RESID(-2)^2	0.199839	0.100202	1.994368	0.0461
RESID(-3)^2	-0.018096	0.032020	-0.565150	0.5720
RESID(-4)^2	0.135498	0.097645	1.387657	0.1652
RESID(-5)^2	0.142758	0.068628	2.080160	0.0375
@QUARTER=2	-3.564372	1.536017	-2.320528	0.0203
@QUARTER=3	-4.325761	1.594279	-2.713302	0.0067
@QUARTER=4	-5.097462	1.511509	-3.372433	0.0007
R-squared	0.346455	Mean dependent var	-0.042844	
Adjusted R-squared	0.326138	S.D. dependent var	2.805045	
S.E. of regression	2.302637	Akaike info criterion	4.221734	
Sum squared resid	1023.312	Schwarz criterion	4.485600	
Log likelihood	-406.1734	Hannan-Quinn criter.	4.328517	
Durbin-Watson stat	1.853424			
Inverted AR Roots	-.04-.56i	-.04+.56i	-.67	

Date: 03/21/12 Time: 21:21
 Sample: 1951Q1 2000Q4
 Included observations: 200
 Q-statistic probabilities adjusted for 3 ARMA term(s)

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1	0.017	0.017	0.0592
		2	0.036	0.036	0.3262
		3	0.176	0.175	6.6481
		4	0.039	0.034	6.9569
		5	-0.028	-0.042	7.1144
		6	0.016	-0.018	7.1649
		7	-0.077	-0.090	8.4033
		8	-0.140	-0.135	12.552
		9	-0.123	-0.122	15.747
		10	-0.097	-0.068	17.749
		11	-0.068	-0.009	18.729
		12	0.009	0.068	18.745

But this will not help you much.



Series: Standardized Residuals	
Sample 1951Q1 2000Q4	
Observations 200	
Mean	0.079501
Median	0.090113
Maximum	3.100286
Minimum	-2.608892
Std. Dev.	1.012874
Skewness	-0.083539
Kurtosis	2.829529
Jarque-Bera	0.474793
Probability	0.788679

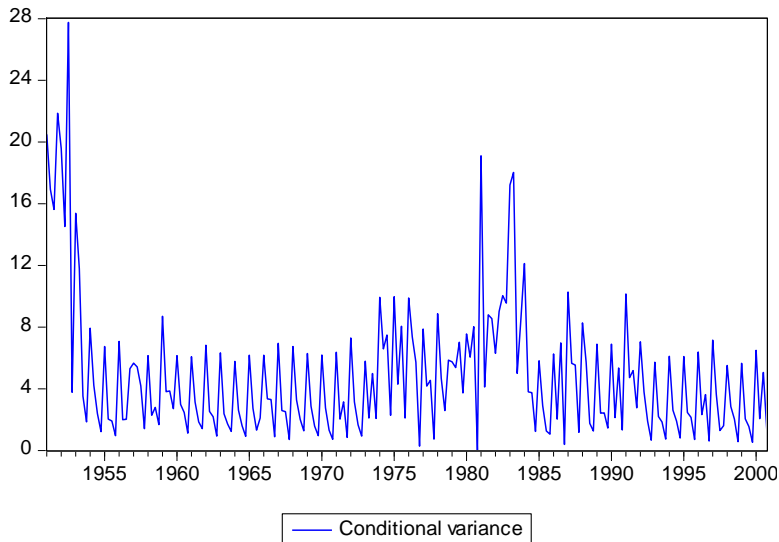
The residuals seem normally distributed though.

We can also check if there is any ARCH type heteroscedasticity remaining in the residuals:

Heteroskedasticity Test: ARCH			
F-statistic	0.694496	Prob. F(4,191)	0.5966
Obs*R-squared	2.809837	Prob. Chi-Square(4)	0.5901

Test Equation:				
Dependent Variable: WGT_RESID^2				
Method: Least Squares				
Date: 03/21/12 Time: 21:22				
Sample (adjusted): 1952Q1 2000Q4				
Included observations: 196 after adjustments				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.910731	0.169985	5.357713	0.0000
WGT_RESID^2(-1)	0.061617	0.072361	0.851529	0.3955
WGT_RESID^2(-2)	-0.041867	0.072151	-0.580267	0.5624
WGT_RESID^2(-3)	0.099590	0.071469	1.393472	0.1651
WGT_RESID^2(-4)	-0.021861	0.071795	-0.304487	0.7611
R-squared	0.014336	Mean dependent var	1.009918	
Adjusted R-squared	-0.006306	S.D. dependent var	1.379002	
S.E. of regression	1.383344	Akaike info criterion	3.512064	
Sum squared resid	365.5053	Schwarz criterion	3.595689	
Log likelihood	-339.1822	Hannan-Quinn criter.	3.545919	
F-statistic	0.694496	Durbin-Watson stat	1.973177	
Prob(F-statistic)	0.596613			

Our estimate for the residual variance can also be graphed:



Showing that the volatility of inflation was especially large during the first half of the 1980s and the beginning of our sample, the early 1950s.

7.d GARCH: motivation and estimation

Bollerslev introduced a more general formation of the conditional heteroscedasticity models. The GARCH (p,m) model looks as follows:

$$\sigma_{u_t}^2 = h_t = \alpha_0 + \sum_{i=1}^p \gamma_i \sigma_{u_{t-i}}^2 + \sum_{j=1}^m \alpha_j u_{t-j}^2$$

since $e_t = u_t^2 - \sigma_{u_t}^2$, we can rewrite this as:

$$u_t^2 = \alpha_0 + \sum_{i=1}^p \gamma_i \sigma_{u_{t-i}}^2 + \sum_{j=1}^m \alpha_j u_{t-j}^2 + e_t = \alpha_0 + \sum_{i=1}^p \gamma_i u_{t-i}^2 + \sum_{j=1}^m \alpha_j u_{t-j}^2 + e_t - \sum_{i=1}^p \gamma_i e_{t-i}$$

which is an ARMA(max(p,m),p) representation of the square residual.

In case of a GARCH(1,1) we assume that:

$$\sigma_{u_t}^2 = \alpha_0 + \gamma_1 \sigma_{u_{t-1}}^2 + \alpha_1 u_{t-1}^2, \text{ which is equivalent with believing that}$$

$$u_t^2 = \alpha_0 + (\alpha_1 + \gamma_1) u_{t-1}^2 + e_t - \gamma_1 e_{t-1} \text{ or}$$

$$\sigma_{u_t}^2 = \alpha_0 + (\gamma_1 + \alpha_1) \sigma_{u_{t-1}}^2 + \alpha_1 e_{t-1}$$

Obviously, this model is stationary if $\alpha_1 + \gamma_1 < 1$. If this is not the case, the residual variance (or the volatility) is not finite.

A GARCH model is usually estimated by a ML procedure.

The above model can be re-estimated as a GARCH model as follows:

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/22/12 Time: 14:48
 Sample (adjusted): 1951Q1 2000Q4
 Included observations: 200 after adjustments
 Convergence achieved after 40 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(8) + C(9)*RESID(-1)^2 + C(10)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.901551	0.450929	1.999320	0.0456
@QUARTER=2	-0.495605	0.849858	-0.583161	0.5598
@QUARTER=3	-1.380489	0.443816	-3.110496	0.0019
@QUARTER=4	-1.717142	0.697461	-2.461989	0.0138
AR(1)	-0.726351	0.081088	-8.957599	0.0000
AR(2)	-0.334132	0.095639	-3.493673	0.0005
AR(3)	-0.199581	0.081265	-2.455929	0.0141

Variance Equation				
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	4.564941	1.577484	2.893811	0.0038
RESID(-1)^2	0.115425	0.048571	2.376422	0.0175
GARCH(-1)	0.775049	0.074079	10.46241	0.0000
@QUARTER=2	-7.262906	2.784994	-2.607871	0.0091
@QUARTER=3	-4.257227	1.747629	-2.436002	0.0149
@QUARTER=4	-5.123921	1.823769	-2.809522	0.0050

R-squared	0.344733	Mean dependent var	-0.042844
Adjusted R-squared	0.324362	S.D. dependent var	2.805045
S.E. of regression	2.305669	Akaike info criterion	4.256644
Sum squared resid	1026.009	Schwarz criterion	4.471035
Log likelihood	-412.6644	Hannan-Quinn criter.	4.343405
Durbin-Watson stat	1.893810		

Inverted AR Roots	-.03+.54i	-.03-.54i	-.67
-------------------	-----------	-----------	------

The Akaike Information Criterion suggest that the ARCH(5) fits the data better than a GARCH(1,1) but the Schwarz criterion suggest that the GARCH is the better model. A GARCH(2,1) will perform even better:

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/22/12 Time: 14:50
 Sample (adjusted): 1951Q1 2000Q4
 Included observations: 200 after adjustments
 Convergence achieved after 114 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(8) + C(9)*RESID(-1)^2 + C(10)*GARCH(-1) + C(11)*GARCH(-2)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.921125	0.415876	2.214901	0.0268
@QUARTER=2	-0.417032	0.741255	-0.562603	0.5737
@QUARTER=3	-1.389687	0.479989	-2.895245	0.0038
@QUARTER=4	-2.002786	0.639554	-3.131537	0.0017
AR(1)	-0.708214	0.072621	-9.752238	0.0000
AR(2)	-0.359943	0.081921	-4.393797	0.0000
AR(3)	-0.167350	0.077555	-2.157806	0.0309

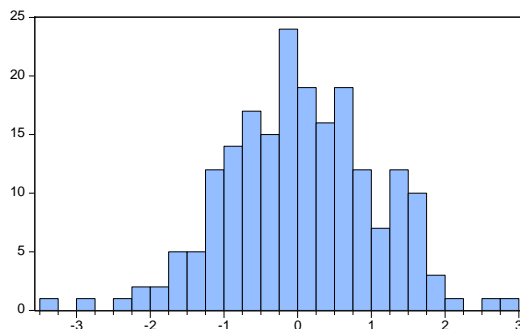
Variance Equation				
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	5.615685	1.386307	4.050822	0.0001
RESID(-1)^2	0.036350	0.017507	2.076264	0.0379
GARCH(-1)	1.549517	0.098583	15.71782	0.0000
GARCH(-2)	-0.655416	0.072552	-9.033748	0.0000
@QUARTER=2	-11.95909	3.451115	-3.465283	0.0005
@QUARTER=3	-2.145363	0.571290	-3.755293	0.0002
@QUARTER=4	-7.281530	2.377345	-3.062884	0.0022

R-squared	0.361257	Mean dependent var	-0.042844
Adjusted R-squared	0.341400	S.D. dependent var	2.805045
S.E. of regression	2.276412	Akaike info criterion	4.246066
Sum squared resid	1000.136	Schwarz criterion	4.476948
Log likelihood	-410.6066	Hannan-Quinn criter.	4.339500
Durbin-Watson stat	1.895977		

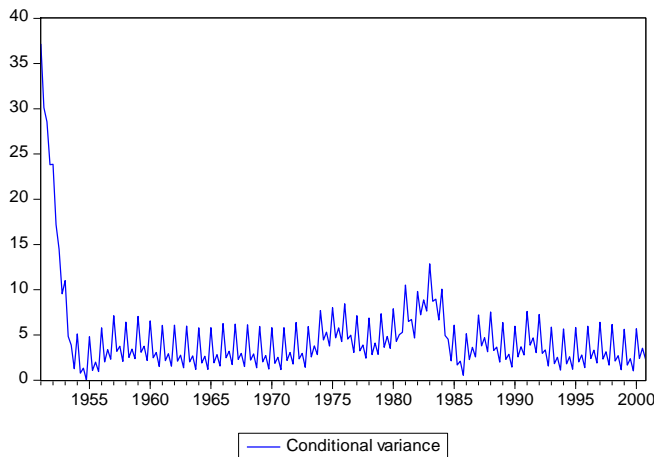
Inverted AR Roots	-.06-.53i	-.06+.53i	-.58
-------------------	-----------	-----------	------

Date: 03/22/12 Time: 14:53
 Sample: 1951Q1 2000Q4
 Included observations: 200
 Q-statistic probabilities adjusted for 3 ARMA term(s)

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
		1	0.019	0.019	0.0709	
		2	0.036	0.035	0.3319	
		3	0.109	0.108	2.7580	
		4	0.065	0.061	3.6206	0.057
		5	-0.014	-0.024	3.6641	0.160
		6	-0.015	-0.031	3.7114	0.294
		7	-0.109	-0.123	6.1991	0.185
		8	-0.122	-0.123	9.3169	0.097
		9	-0.108	-0.098	11.780	0.067
		10	-0.087	-0.057	13.402	0.063
		11	-0.088	-0.044	15.049	0.058
		12	0.047	0.091	15.520	0.078



Series: Standardized Residuals	
Sample 1951Q1 2000Q4	
Observations 200	
Mean	0.031804
Median	0.017190
Maximum	2.851424
Minimum	-3.277575
Std. Dev.	1.008829
Skewness	-0.107196
Kurtosis	3.226851
Jarque-Bera	0.811880
Probability	0.666350



7.e. Further extensions of the methodology

There are a lot of possible extensions of the conditional heteroscedasticity methodology.

Interrelation between variance and level (GARCH-M)

We can assume that higher volatility has an effect on the level. This means a small modification of the level equation:

$$y_t = \beta_0 + \sum_{i=1}^p \beta_i x_{it} + \lambda \sigma_{u_t} + u_t$$

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/22/12 Time: 15:03
 Sample (adjusted): 1951Q1 2000Q4
 Included observations: 200 after adjustments
 Convergence achieved after 46 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(9) + C(10)*RESID(-1)^2 + C(11)*GARCH(-1) + C(12)*GARCH(-2)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	-0.514075	0.238460	-2.155808	0.0311
C	2.618890	1.034634	2.531224	0.0114
@QUARTER=2	-1.431268	1.086876	-1.316864	0.1879
@QUARTER=3	-1.864605	0.564667	-3.302132	0.0010
@QUARTER=4	-2.703709	0.967540	-2.794416	0.0052
AR(1)	-0.753192	0.070975	-10.61213	0.0000
AR(2)	-0.359499	0.082642	-4.350101	0.0000
AR(3)	-0.198358	0.076475	-2.593758	0.0095

Variance Equation				
C	5.556949	1.701908	3.265129	0.0011
RESID(-1)^2	0.051289	0.034439	1.489281	0.1364
GARCH(-1)	1.460726	0.224192	6.515517	0.0000
GARCH(-2)	-0.584372	0.172579	-3.386115	0.0007
@QUARTER=2	-11.08967	4.074413	-2.721784	0.0065
@QUARTER=3	-3.162071	1.156543	-2.734073	0.0063
@QUARTER=4	-6.892989	2.724430	-2.530067	0.0114

R-squared	0.381702	Mean dependent var	-0.042844
Adjusted R-squared	0.359160	S.D. dependent var	2.805045
S.E. of regression	2.245509	Akaike info criterion	4.225352
Sum squared resid	968.1236	Schwarz criterion	4.472726
Log likelihood	-407.5352	Hannan-Quinn criter.	4.325460
Durbin-Watson stat	1.944235		

Inverted AR Roots			
	-.05-.55i	-.05+.55i	-.66

We find here that in periods with high volatility inflation tends to be lower.

Threshold ARCH and GARCH (T-ARCH/T-GARCH)

It is also possible that the previous period's volatility has asymmetrical effect on today's volatility depending on the movement of the dependent variable. In other words, if we had a

positive shock in inflation (u_{t-1} is positive) than it is going to have a bigger impact on volatility now than a negative shock. This requires that we adjust the variance equation a bit:

$$\sigma_{u_t}^2 = \alpha_0 + \gamma_1 \sigma_{u_{t-1}}^2 + \alpha_1 u_{t-1}^2 + \alpha_2 (I(u_{t-1} < 1) \cdot u_{t-1}^2)$$

Where $I(u_{t-1} < 1)$ is an indicator variable (a dummy) taking the value one only if the previous period's residual was negative.

Dependent Variable: D(INFL)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/22/12 Time: 15:36
Sample (adjusted): 1951Q1 2000Q4
Included observations: 200 after adjustments
Convergence achieved after 64 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(8) + C(9)*RESID(-1)^2 + C(10)*RESID(-1)^2*(RESID(-1)<0) + C(11)*GARCH(-1) + C(12)*GARCH(-2)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	1.121377	0.455734	2.460595	0.0139
@QUARTER=2	-0.798675	0.814223	-0.980905	0.3266
@QUARTER=3	-1.552376	0.434964	-3.568975	0.0004
@QUARTER=4	-2.024757	0.679984	-2.977657	0.0029
AR(1)	-0.760761	0.065695	-11.58027	0.0000
AR(2)	-0.398581	0.080791	-4.933476	0.0000
AR(3)	-0.203313	0.075691	-2.686103	0.0072

Variance Equation				
C	5.233534	1.494539	3.501773	0.0005
RESID(-1)^2	0.115808	0.050772	2.280963	0.0226
RESID(-1)^2*(RESID(-1)<0)	-0.094008	0.053062	-1.771658	0.0765
GARCH(-1)	1.483559	0.103049	14.39665	0.0000
GARCH(-2)	-0.595250	0.082853	-7.184391	0.0000
@QUARTER=2	-11.46700	3.418010	-3.354877	0.0008
@QUARTER=3	-1.893425	1.042680	-1.815922	0.0694
@QUARTER=4	-6.769010	2.266765	-2.960081	0.0031

R-squared	0.350893	Mean dependent var	-0.042844
Adjusted R-squared	0.330714	S.D. dependent var	2.805045
S.E. of regression	2.294805	Akaike info criterion	4.232063
Sum squared resid	1016.363	Schwarz criterion	4.479437
Log likelihood	-408.2063	Hannan-Quinn criter.	4.332172
Durbin-Watson stat	1.797498		

Inverted AR Roots	-.06-.56i	-.06+.56i	-.64
-------------------	-----------	-----------	------

The results suggest that the effect of a shock on the volatility of inflation does depend on the direction of the shock. Negative shock to inflation has a much lower effect on volatility, than positive shocks. Actually, with a Wald-test we can even find out that a negative shock has no impact on volatility at all while a positive shock does.

Wald Test:
Equation: Untitled

Test Statistic	Value	df	Probability
t-statistic	1.385036	185	0.1677
F-statistic	1.918325	(1, 185)	0.1677
Chi-square	1.918325	1	0.1660

Null Hypothesis: C(9) + C(10)=0
Null Hypothesis Summary:

Normalized Restriction (= 0)	Value	Std. Err.
C(9) + C(10)	0.021800	0.015740

Restrictions are linear in coefficients.

Exponential GARCH (EGARCH)

The variance equation is modified as follows (EGARCH(1,1):

$$\ln \sigma_{u_t}^2 = \alpha_0 + \gamma_1 \ln \sigma_{u_{t-1}}^2 + \alpha_1 \left| \frac{u_{t-1}}{\sigma_{u_{t-1}}} \right|$$

or if you expect an asymmetric effect of shocks:

$$\ln \sigma_{u_t}^2 = \alpha_0 + \gamma_1 \ln \sigma_{u_{t-1}}^2 + \alpha_1 \left| \frac{u_{t-1}}{\sigma_{u_{t-1}}} \right| + \lambda_1 \alpha_1 \frac{u_{t-1}}{\sigma_{u_{t-1}}}$$

where, if $\lambda_1 \neq 0$, there is an asymmetry. Additional advantage is the you have can only have positive estimated values of conditional volatility.

Dependent Variable: D(INFL)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/26/12 Time: 11:56
 Sample (adjusted): 1951Q1 2000Q4
 Included observations: 200 after adjustments
 Convergence achieved after 44 iterations
 Presample variance: backcast (parameter = 0.7)
 LOG(GARCH) = C(8) + C(9)*ABS(RESID(-1))/@SQRT(GARCH(-1))) + C(10)
 *RESID(-1)/@SQRT(GARCH(-1)) + C(11)*LOG(GARCH(-1))

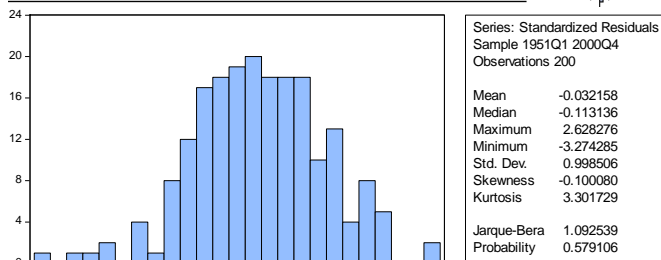
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	1.630831	0.486514	3.352076	0.0008
@QUARTER=2	-1.653426	0.900873	-1.835358	0.0665
@QUARTER=3	-1.900318	0.449841	-4.224419	0.0000
@QUARTER=4	-2.767523	0.737193	-3.754138	0.0002
AR(1)	-0.737258	0.077318	-9.535454	0.0000
AR(2)	-0.376033	0.081053	-4.639325	0.0000
AR(3)	-0.249981	0.076194	-3.280838	0.0010

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C(8)	0.954632	0.250110	3.816849	0.0001
C(9)	0.313581	0.120152	2.609876	0.0091
C(10)	0.175131	0.081945	2.137180	0.0326
C(11)	0.944324	0.037607	25.11023	0.0000
C(12)	-1.822741	0.454974	-4.006254	0.0001
C(13)	-1.186893	0.396441	-2.993872	0.0028
C(14)	-1.555944	0.453715	-3.429339	0.0006

	Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
1	-0.002	-0.002	0.0010			
2	0.009	0.009	0.0183			
3	0.157	0.157	5.0946			
4	0.017	0.018	5.1548	0.023		
5	-0.005	-0.008	5.1600	0.076		
6	-0.009	-0.035	5.1785	0.159		
7	-0.114	-0.123	7.9163	0.095		
8	-0.099	-0.103	9.9911	0.075		
9	-0.110	-0.109	12.562	0.051		
10	-0.117	-0.088	15.488	0.030		
11	-0.123	-0.097	18.715	0.016		
12	0.036	0.071	18.988	0.025		
13	-0.016	0.023	19.045	0.040		
14	-0.043	-0.017	19.440	0.054		
15	-0.052	-0.094	20.019	0.067		
16	0.106	0.066	22.483	0.048		
17	0.043	0.011	22.898	0.062		
18	0.045	0.019	23.350	0.077		
19	-0.108	-0.176	25.953	0.055		
20	0.106	0.069	28.487	0.040		
21	0.063	0.042	29.392	0.044		

	Mean dependent var	S.D. dependent var
R-squared	0.338050	2.805045
Adjusted R-squared	0.317471	4.239608
S.E. of regression	2.317397	4.470490
Sum squared resid	1036.474	4.333042
Log likelihood	-409.9608	
Durbin-Watson stat	1.862751	

	Mean dependent var	S.D. dependent var
Inverted AR Roots	-.02+.59i	-.71



7.f. Some applications

Forecasting risk:

We take the Dow Jones index at close between Jan 1990 and Apr 2011 (dataset is dj.wf1). This is a monthly data, normally we would use high frequency data (daily or weekly) but this will do now. The log difference of the index is a kind of weighted monthly returns on the stocks traded at NYSE. The returns are stationary:

Null Hypothesis: R is stationary
 Exogenous: Constant, Linear Trend
 Bandwidth: 3 (Newey-West automatic) using Bartlett kernel

	LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic	0.061448
Asymptotic critical values*:	
1% level	0.216000
5% level	0.146000
10% level	0.119000

*Kwiatkowski-Phillips-Schmidt-Shin (1992, Table 1)

Residual variance (no correction)	0.001845
HAC corrected variance (Bartlett kernel)	0.001858

And has no serial correlations:

Date: 03/22/12 Time: 16:38
 Sample: 1990M01 2011M04
 Included observations: 255

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1 0.026	0.026	0.1711	0.679
		2 -0.025	-0.026	0.3388	0.844
		3 0.024	0.025	0.4850	0.922
		4 0.052	0.051	1.2033	0.878
		5 0.020	0.018	1.3033	0.935
		6 -0.099	-0.099	3.9019	0.690
		7 0.085	0.090	5.8321	0.559
		8 0.054	0.042	6.6105	0.579
		9 -0.026	-0.023	6.7963	0.658
		10 -0.054	-0.047	7.5858	0.669
		11 0.037	0.034	7.9548	0.717
		12 0.122	0.104	11.978	0.447
		13 -0.107	-0.097	15.055	0.304
		14 -0.037	-0.023	15.437	0.349
		15 -0.018	-0.037	15.523	0.414
		16 0.043	0.034	16.041	0.450
		17 -0.008	0.010	16.057	0.520
		18 0.005	0.008	16.004	0.524

This seems to nicely confirm the weak form of the efficient market hypothesis.
 The distribution of the returns is not normal though:

Empirical Distribution Test for R
 Hypothesis: Normal
 Date: 03/22/12 Time: 16:39
 Sample (adjusted): 1990M02 2011M04
 Included observations: 255 after adjustments

Method	Value	Adj. Value	Probability
Lilliefors (D)	0.074351	NA	0.0016
Cramer-von Mises (W2)	0.318493	0.319118	0.0002
Watson (U2)	0.257271	0.257775	0.0005
Anderson-Darling (A2)	1.889484	1.895107	0.0001

Method: Maximum Likelihood - d.f. corrected (Exact Solution)

Parameter	Value	Std. Error	z-Statistic	Prob.
MU	0.006122	0.002703	2.264783	0.0235
SIGMA	0.043166	0.001915	22.53886	0.0000

Log likelihood	440.0598	Mean dependent var.	0.006122
No. of Coefficients	2	S.D. dependent var.	0.043166

Heteroskedasticity Test: ARCH

F-statistic	3.551272	Prob. F(4,246)	0.0077
Obs*R-squared	13.70257	Prob. Chi-Square(4)	0.0083

Test Equation:
 Dependent Variable: RESID^2
 Method: Least Squares
 Date: 03/22/12 Time: 16:40
 Sample (adjusted): 1990M06 2011M04
 Included observations: 251 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.001143	0.000298	3.838275	0.0002
RESID^2(-1)	0.107917	0.062913	1.715350	0.0875
RESID^2(-2)	0.031343	0.063025	0.497305	0.6194
RESID^2(-3)	0.093404	0.063026	1.481982	0.1396
RESID^2(-4)	0.148028	0.062921	2.352603	0.0194

R-squared	0.054592	Mean dependent var	0.001859
Adjusted R-squared	0.039219	S.D. dependent var	0.003529
S.E. of regression	0.003459	Akaike info criterion	-8.475823
Sum squared resid	0.002944	Schwarz criterion	-8.405595

And we find evidence for an ARCH type structure in the volatility of the returns.
 An GARCH(1,1) model seems to fit the data nicely:

Dependent Variable: R
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/22/12 Time: 16:43
 Sample (adjusted): 1990M02 2011M04
 Included observations: 255 after adjustments
 Convergence achieved after 17 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Dependent Variable: R
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/22/12 Time: 16:44
 Sample (adjusted): 1990M02 2011M04
 Included observations: 255 after adjustments
 Convergence achieved after 19 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-1)^2*(RESID(-1)<0) + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.007179	0.002576	2.786951	0.0053

Variance Equation				
C	6.86E-05	4.64E-05	1.479173	0.1391
RESID(-1)^2	0.132155	0.040863	3.234094	0.0012
GARCH(-1)	0.839703	0.040907	20.52709	0.0000

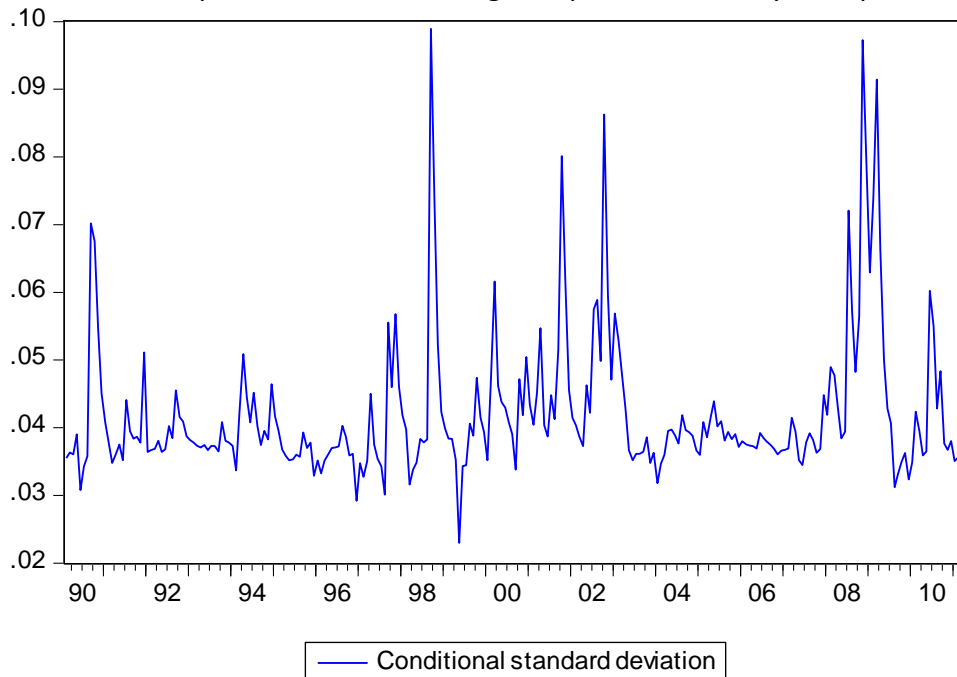
R-squared	-0.000601	Mean dependent var	0.006122
Adjusted R-squared	-0.000601	S.D. dependent var	0.043166
S.E. of regression	0.043179	Akaike info criterion	-3.511108
Sum squared resid	0.473564	Schwarz criterion	-3.455559
Log likelihood	451.8663	Hannan-Quinn criter.	-3.488764
Durbin-Watson stat	1.947148		

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.006000	0.002747	2.184242	0.0289

Variance Equation				
C	0.000697	0.000276	2.529705	0.0114
RESID(-1)^2	-0.094067	0.054311	-1.732026	0.0833
RESID(-1)^2*(RESID(-1)<0)	0.383116	0.154018	2.487472	0.0129
GARCH(-1)	0.497961	0.201954	2.465712	0.0137

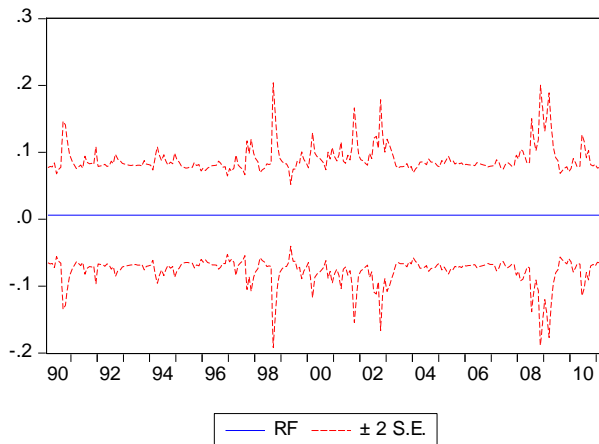
R-squared	-0.000008	Mean dependent var	0.006122
Adjusted R-squared	-0.000008	S.D. dependent var	0.043166
S.E. of regression	0.043166	Akaike info criterion	-3.512065
Sum squared resid	0.473283	Schwarz criterion	-3.442628
Log likelihood	452.7883	Hannan-Quinn criter.	-3.484135
Durbin-Watson stat	1.948304		

We find that a TGARCH model is even better, and negative shocks in returns seem to have had a bigger impact on volatility than positive shocks. This is referred to as leverage effect in the literature: price falls can have larger impact on volatility than price increases.

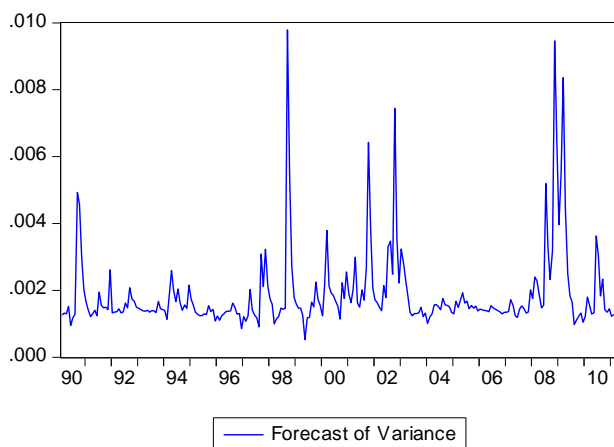


We can observe that certain periods were especially volatile, like the first half of 1998 or, not surprisingly, mid-2008.

One thing we can do with an ARCH/GARCH model is to forecast how risky the next period is going to be. We can use a static forecast to see how good the model would have performed in one period ahead forecasts:



Forecast: RF	
Actual: R	
Forecast sample: 1990M01 2011M04	
Adjusted sample: 1990M02 2011M04	
Included observations: 255	
Root Mean Squared Error	0.043081
Mean Absolute Error	0.032276
Mean Abs. Percent Error	177.3381
Theil Inequality Coefficient	0.870079
Bias Proportion	0.000008
Variance Proportion	NA
Covariance Proportion	NA



Looking for trends in market performance:

In economic history the working of markets has become a focal point. Event hough this is often labeled as market efficiency, we should rather distinguish it from standard market efficiency (as suggested by Fama 1965)³. Normally economic historians look at the volatility of prices as a measure of the degree of integration and generally the efficiency of the markets in coping with risks.

As long as this exercises is done on cross-sectional data (that is, one looks at prices within the same period but from different places) we have nothing to add here. Still, it became increasingly popular to look not only on the spatial but also on the dynamic aspect of price volatility, where some special time-series problems may appear. For a detailed description you can consult Földvári and Van Leeuwen (2011).⁴

In case of cross sectional data, it is a standard exercise to calculate the Coefficient of Variation (CV) of the observed prices. The CV is the ratio of the standard deviation and the mean. If we have reason to believe that your observations come from the same probability distribution, that is, there exist a finite mean and standard deviation of them, this method

³ Fama, E. F. (1965) The Behavior of Stock-Market Prices. The Journal of Business, Vol. 38, No. 1., 34-105.

⁴ Földvári, P. and Van Leeuwen, B. (2011) Conditional heteroscedasticity in historical commodity price series Cliometrica, 5(2):165-186).

works fine. For example, you can calculate the CV for the price of grain for 50 cities in Europe in 1800 and in 1850 and then compare the two CVs.

Recently more authors started to look at the CV measured over time. So, they take single geographical unit, say a single city, and calculate the CV of the prices for a given period. Then they repeat this for another interval of time and they draw conclusions based on the observed differences. In the cited article we argue that this is incorrect. To see why, we need to accept that in most cases prices tend to exhibit trend-like behavior. This can be due to trend-stationarity but also if prices are random-walks with drift. The latter is more in line with the classical understanding of the efficient markets, but in case of historical prices you can find stationary prices as well.

Let us assume that the logarithm of a price (p_t) is a random-walk with drift process:

$$\ln p_t = \alpha + \ln p_{t-1} + \varepsilon_t$$

Using logarithm is not only due to some nice properties of taking logarithm, but because the standard deviation of the log prices is very close to the CV of the level of prices. So what we can find out about the standard deviation of the log prices is also true for the CV of the prices, without logarithmization.

We already know that the above process can be rewritten as:

$\ln p_t = \ln p_0 + \alpha t + \sum_{i=1}^t \varepsilon_i$ if $\varepsilon_i \sim WN(0, \sigma_\varepsilon^2)$ we would obtain the well-known relationship:

$$\sigma_{\ln p_t}^2 = t\sigma_\varepsilon^2 \text{ or } \sigma_{\ln p_t} = \sqrt{t}\sigma_\varepsilon \text{ where we already used the idea that } E(\ln p_t) = \ln p_0 + \alpha t .$$

Obviously, both moments (expected value and variance) depends on time. That is, there is no unique, finite mean or variance. The method of Coefficient of Variation depends on the assumption that you have a variance and a mean that is constant over time. This is obviously not right.

So what should we expect? The longer the sample, the higher the variance gets. This has nothing to do with any "efficiency" of the markets whatsoever, or a detrimental process in integration of markets. This comes simply because the underlying prices are not mean-reverting and this inflates variance.

One solution is to difference prices before calculating their variance. This is a possible way to go, but it will only work if your price series are indeed classic random-walk processes. If not, because the DGP looks like this.

$\ln p_t = \alpha_0 + \alpha_1 \ln p_{t-1} + \alpha_2 \ln p_{t-2} + \varepsilon_t$, first-differencing will yield the following:

$\Delta \ln p_t = \alpha_0 + (\alpha_1 - 1) \ln p_{t-1} + \alpha_2 \ln p_{t-2} + \varepsilon_t$ so your estimated volatility of log p will have more than what you really wished for:

$$\sigma_{\Delta \ln p_t}^2 = (\alpha_1 - 1)^2 \sigma_{\ln p_{t-1}}^2 + \alpha_2^2 \sigma_{\ln p_{t-2}}^2 + \sigma_{\varepsilon_t}^2$$

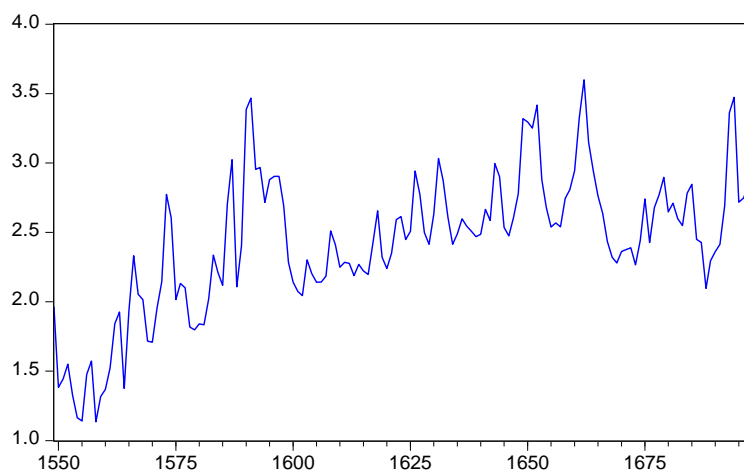
For this reason the first step should be a modeling of the expected value of the prices (estimating a level equation) than you can further estimate the residual variance (variance equation). In other words, you should rather trust an ARCH/GARCH framework once you would like to learn something about the time trend of the volatility of prices. The volatility can be thought to be a combination of purely random shocks (θ) that is assumed to be a white noise process and a multiplier that reflect the market's ability to cope with this shocks (λ_t).

$$\varepsilon_t = \theta_t \sqrt{\lambda_t}, \text{ where } \theta_t \sim WN(0, \sigma_\theta^2)$$

If we find lambda to decrease over time, we have evidence for a better performance of the market to cope with the unexpected.

To test this we have now the price of wheat in Paris 1549-1697.

LNWHEAT



There is an obvious upward trend in the data. If you used the standard methodology you would find something like this:

sample	std deviation of log wheat (roughly the CV)	t
1549-1697	0.503	149
1549-1580	0.406	32
1581-1640	0.333	60
1641-1697	0.400	57

The most striking finding is that the volatility of the whole sample, that we would expect to be some kind of average of the volatility of the sub-samples, is actually higher than any of the sub-samples. The reason is obvious: the primitive method will yield biased estimates. If one still believed in the above results, one would come to the conclusion that there was some improvement in terms of volatility during the first half of the 17th century, but later this turned to the opposite. One way is to take the first differences:

Dependent Variable: D(LNWHEAT)
 Method: Least Squares
 Date: 03/24/12 Time: 10:18
 Sample (adjusted): 1550 1697
 Included observations: 148 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.005856	0.022222	0.263512	0.7925
R-squared	0.000000	Mean dependent var		0.005856
Adjusted R-squared	0.000000	S.D. dependent var		0.270338
S.E. of regression	0.270338	Akaike info criterion		0.228449
Sum squared resid	10.74318	Schwarz criterion		0.248700
Log likelihood	-15.90523	Hannan-Quinn criter.		0.236677
Durbin-Watson stat	1.913972			

The estimated standard deviation is now 0.270, much lower than with the above method. Still, even this can be misleading as this residual variance may contain a lot of information that could not really be brought into connection with any kind of market performance. So we find the best ARMA model. The big question is whether we should difference the log prices or not:

Null Hypothesis: LNWHEAT has a unit root
 Exogenous: Constant, Linear Trend
 Lag length: 2 (Spectral GLS-detrended AR based on SIC, maxlag=13)
 Sample: 1549 1697
 Included observations: 149

	MZa	MZt	MSB	MPT
Ng-Perron test statistics	-20.2310	-3.17892	0.15713	4.51400
Asymptotic critical values*:				
1%	-23.8000	-3.42000	0.14300	4.03000
5%	-17.3000	-2.91000	0.16800	5.48000
10%	-14.2000	-2.62000	0.18500	6.67000

*Ng-Perron (2001, Table 1)

HAC corrected variance (Spectral GLS-detrended AR) 0.042528

The best known unit-root test (Ng-Perron) cannot help us out here as it can reject the unit root hypothesis at 5% but not at 1% (MZa and MZt) and it can reject stationarity only at 1% (MSB and MPT).

Luckily, it does not make much difference if we just take the first difference of the prices independently whether it has a unit-root or not. Nevertheless, with historical data you should be careful not to difference unless it is needed since the ratio of measurement errors to signal may increase as a result of differencing. The seemingly best model has only a second-order autoregressive term:

Dependent Variable: D(LNWHEAT)
 Method: Least Squares
 Date: 03/24/12 Time: 10:25
 Sample (adjusted): 1552 1697
 Included observations: 146 after adjustments
 Convergence achieved after 3 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.008310	0.014876	0.558614	0.5773
AR(2)	-0.379175	0.075655	-5.011920	0.0000
R-squared	0.148530	Mean dependent var		0.009510
Adjusted R-squared	0.142617	S.D. dependent var		0.267713
S.E. of regression	0.247889	Akaike info criterion		0.061929
Sum squared resid	8.848621	Schwarz criterion		0.102801
Log likelihood	-2.520838	Hannan-Quinn criter.		0.078536
F-statistic	25.11935	Durbin-Watson stat		1.888415
Prob(F-statistic)	0.000002			

But even this has already reduced the residual standard deviation to 0.248, so our fears that taking simply first-difference may not be enough (can be a good approximation, though) is justified. There is no apparent serial correlation in the residual:

Date: 03/24/12 Time: 10:28
 Sample: 1552 1697
 Included observations: 146
 Q-statistic probabilities adjusted for 1 ARMA term(s)

Date: 03/24/12 Time: 10:28
 Sample: 1552 1697
 Included observations: 146
 Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
0.002	0.002	1	0.052	0.052	0.4088	0.002	0.002	1	0.021	0.021	0.0659
0.001	0.001	2	-0.048	-0.051	0.7519	0.386	0.001	0.227	0.227	7.7933	0.005
0.001	0.001	3	-0.028	-0.023	0.8731	0.646	0.001	-0.112	-0.127	9.6916	0.008
0.001	0.001	4	-0.116	-0.116	2.9075	0.406	0.001	0.166	0.130	13.871	0.003
0.001	0.001	5	-0.117	-0.110	5.0157	0.286	0.001	0.002	0.045	13.871	0.008
0.001	0.001	6	0.003	0.001	5.0170	0.414	0.001	-0.007	-0.094	13.878	0.016
0.001	0.001	7	-0.121	-0.144	7.3107	0.293	0.001	-0.060	-0.030	14.435	0.025
0.001	0.001	8	-0.024	-0.034	7.4036	0.388	0.001	0.032	0.045	14.592	0.042
0.001	0.001	9	-0.048	-0.094	7.7718	0.456	0.001	0.024	0.020	14.680	0.066
0.001	0.001	10	0.045	0.025	8.0985	0.524	0.001	-0.019	-0.037	14.736	0.098
0.001	0.001	11	-0.004	-0.053	8.1012	0.619	0.001	0.024	0.046	14.828	0.138
0.001	0.001	12	-0.053	-0.096	8.5496	0.663	0.001	-0.005	0.000	14.832	0.190
0.001	0.001	13	0.036	0.016	8.7561	0.724	0.001	0.095	0.065	16.301	0.178
0.001	0.001	14	-0.016	-0.064	8.7977	0.788	0.001	-0.043	-0.038	16.604	0.218
0.001	0.001	15	-0.096	-0.110	10.321	0.738	0.001	0.101	0.076	18.283	0.194
0.001	0.001	16	-0.071	-0.126	11.148	0.742	0.001	-0.040	-0.017	18.553	0.235
0.001	0.001	17	0.113	0.096	13.270	0.653	0.001	0.065	-0.005	19.259	0.255
0.001	0.001	18	0.033	-0.014	13.458	0.705	0.001	-0.032	0.011	19.429	0.304

But the correlogram of the squared residuals (right) indicate some ARCH type behavior. Now we can turn to a ARCH/GARCH model:

Dependent Variable: D(LNWHEAT)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/24/12 Time: 10:30
 Sample (adjusted): 1552 1697
 Included observations: 146 after adjustments
 Convergence achieved after 18 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*RESID(-2)^2 + C(6)*@TREND

Dependent Variable: D(LNWHEAT)
 Method: ML - ARCH (Marquardt) - Normal distribution
 Date: 03/24/12 Time: 10:32
 Sample (adjusted): 1552 1697
 Included observations: 146 after adjustments
 Convergence achieved after 6 iterations
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*RESID(-2)^2 + C(7)*@TREND

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	-0.746496	0.309706	-2.410336	0.0159
C	0.186667	0.066705	2.798375	0.0051
AR(2)	-0.217332	0.043909	-4.949605	0.0000

Variance Equation				
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.053972	0.013599	3.968784	0.0001
RESID(-1)^2	-0.018400	0.069509	-0.264711	0.7912
RESID(-2)^2	0.200070	0.125419	1.595210	0.1107
@TREND	-6.63E-05	0.000114	-0.582283	0.5604

R-squared	0.130255	Mean dependent var	0.009510
Adjusted R-squared	0.124216	S.D. dependent var	0.267713
S.E. of regression	0.250535	Akaike info criterion	0.069326
Sum squared resid	9.038536	Schwarz criterion	0.191940
Log likelihood	0.939228	Hannan-Quinn criter.	0.119147
Durbin-Watson stat	1.907907		

R-squared	0.174512	Mean dependent var	0.009510
Adjusted R-squared	0.162966	S.D. dependent var	0.267713
S.E. of regression	0.244929	Akaike info criterion	0.025988
Sum squared resid	8.578619	Schwarz criterion	0.169038
Log likelihood	5.102885	Hannan-Quinn criter.	0.084112
Durbin-Watson stat	1.885687		

We introduce a time trend in the variance equation to test if there was any improvement in residual variance, but one we took care of the possibly expandable part of prices, the residual variance does not seem to contain any significant trend any more.